Int. J. Solids Structures Vol. 25, No. 8, pp. 917-945, 1989
Printed in Great Britain.

0020-7683/89 $3.00+ .00

SUBLOADING SURFACE MODEL IN UNCONVENTIONAL PLASTICITY

K. HASHIGUCHI
Department of Agricultural Engineering, Kyushu University, Fukuoka, Japan

(Received 6 April 1988; in revised form 26 November 1988)

Abstract—For engineering problems, the elastoplastic constitutive model has been required, which is applicable to the prediction of cyclic loading behavior for various stress/strain amplitudes. The subloading surface model has been proposed and developed in order to respond to this requirement. The original subloading surface model (or the bounding surface model with a radial mapping) does not assume a yield surface enclosing an elastic domain in which stress rates of any direction do not induce a plastic deformation. Instead, it assumes a normal-yield (or bounding) surface and a subloading surface which always passes through a current stress point in not only loading but also unloading states retaining a geometrical similarity to the normal-yield surface. Thus, it describes a continuous stress rate–strain rate relation in a loading process, bringing about a smooth elastic–plastic transition, and its loading criterion does not require the judgement whether a current stress lies on a yield surface or not. It cannot, however, describe reasonably an induced anisotropy and a hysteresis behavior for a stress change within the normal-yield surface, since the center of similarity of normal-yield and subloading surfaces is fixed or the translation rule is not formulated reasonably. In this paper an exact formulation of this model is presented by deriving a translation rule of the center of similarity and a consistency condition for the subloading surface and by examining the physical meaning of the loading criterion in terms of a strain rate and the associated flow rule concurrently for materials with an anisotropic hardening/softening and without an elastic domain. It is capable of describing an anisotropic hardening/softening, a smooth elastic–plastic transition and a hysteresis behavior including Masing effect, a closed hysteresis loop and a mechanical ratchetting effect consistently. This model is described for metals and is compared with test data of the torsional cyclic loading behavior of stainless steel.

1. INTRODUCTION

A reasonable prediction of inelastic deformation of materials subjected to cyclic loadings is of increasing importance for practical problems in engineering. The conventional theory of plasticity is concerned only with a description of the remarkable plastic deformation in the yield state, ignoring a plastic deformation due to a stress change within the yield surface by assuming its interior to be an elastic domain. The "elastic domain" is defined as a domain in the stress space, in which stress rates of any direction do not induce plastic deformation, i.e. in which only a purely elastic deformation can occur. Obviously, the conventional theory is incapable of predicting cyclic loading behavior for small stress or strain amplitudes. Its extension to the description of plastic deformation induced by the stress change within the yield surface is the inevitable step for the development of plasticity. To this aim, various elastoplastic constitutive models have been proposed since Mroz (1966) proposed the "model of a field of hardening moduli". In the meantime, the author proposed the "subloading surface model" and refined it mathematically (Hashiguchi and Ueno, 1977; Hashiguchi, 1978, 1979, 1980a, 1980b, 1985b). In this extension the state in which a stress lies on the conventional yield surface and the state within the surface are called a "normal-yield state" and a "subyield state", respectively, and the conventional yield surface is called a "normal-yield surface", while it was called a "distinct-yield surface" in the previous paper (Hashiguchi, 1980a). Besides, the "bounding surface" in series of Dafalias' papers (e.g. Dafalias and Popov, 1975; and Dafalias, 1986) is also regarded as the conventional yield surface, since the bounding surface evolves by the isotropic and kinematic hardening rule of the conventional yield surface. The surface in a stress space, on which a stress rate causes a remarkable plastic deformation, has been called a yield surface as seen typically in the perfectly plastic body as the simplest classical idealization. Here, one would have to be deliberate in replacing the term "yield" surface used historically in the theory of plasticity by the term "bounding" surface expressing a geometrical meaning rather than a physical one.

The salient feature of the subloading surface model is the assumption of the "subloading surface" which expands or contracts passing always through a current stress point in not only loading but also unloading states and retaining a geometrical similarity to the normal-yield surface and is the description of a plastic modulus by the ratio of the size of the subloading surface to that of the normal-yield surface. Thus, an elastic domain does not exist and the plastic modulus changes continuously. Then, a continuous stress rate–strain rate relation is described in a loading process, bringing about a smooth elastic–plastic transition, and the loading criterion does not require the judgement whether a stress lies on the yield surface or not since a stress lies always on the subloading surface.

Later on, Dafalias and Herrmann (1980) presented a similar idea of "radial mapping" in which a plastic modulus depends on the ratio of the magnitude of the current stress to that of the conjugate stress on the normal-yield surface but the loading surface is not utilized explicitly, while its mathematical structure is substantially the same as that of the original subloading surface model. They call it a "bounding surface model" as well as the two surface model (Dafalias and Popov, 1975, 1976, 1977). The two surface model assumes a small yield surface, called a "subyield surface" (Hashiguchi, 1981, 1988), which encloses an elastic domain and moves with a plastic deformation within the normal-yield surface, keeping its size constant relatively to the size of the normal-yield surface. Then, the two surface model as well as the multi surface model (model of a field of hardening moduli) (Mróz, 1966, 1967; Iwan, 1967) is regarded as an extension of the kinematic hardening model (Edelman and Drucker, 1951; Ishlinski, 1954; Prager, 1956) to the subyield state. Mróz (1967) stated "we generalize the rules of isotropic and kinematic hardening by introducing the concept of a field of hardening moduli", where he regarded the conventional yield surface exactly as the outermost surface although he did not rename it in particular. On the other hand, the subloading surface expands or contracts with a movement of the current stress point even when a plastic deformation does not occur. Then, the subloading surface model has a different structure from the two surface model although they are occasionally called by the same term "bounding surface model" (Dafalias and Herrmann, 1980; Dafalias, 1986). The bounding surface is the yield surface in the conventional theory as was described before. Since unconventional plasticity models are keeping this surface, one cannot specify models by the term "bounding surface model", and it would not be reasonable to call the two surface model and the radial mapping model by the same term, since they have different structures from each other. On the other hand, the term "subloading surface model" would express concisely the physical feature of this model which is an extension of the conventional theory to the subyield state by assuming the subloading surface within the normal-yield surface.

The subloading surface model or the radial mapping model has been applied widely to the prediction of irreversible deformation of soils (Hashiguchi and Ueno, 1977; Hashiguchi, 1978, 1979, 1980a; Dafalias and Herrmann, 1980, 1982; Aboim and Wroth, 1982; Pande and Pietruszczak, 1982; Dafalias, 1984; Zienkiewicz and Mróz, 1984; Naylor, 1985; Pastor *et al.*, 1985; Zienkiewicz *et al.*, 1985; Anandrajah and Dafalias, 1986; Banerjee and Pan, 1986; Bardet, 1986; Herrmann *et al.*, 1986; Pietruszczak, 1986; Liang *et al.*, 1987; Zienkiewicz and Pastor, 1987), concrete (Fardis *et al.*, 1983; Chen and Buyukozturk, 1985; Yang *et al.*, 1985) and metals (Hashiguchi, 1980a). In these papers, however, the center of similarity of the normal-yield and the subloading surfaces is fixed in the origin of stress space or on the central axis of the normal-yield surface, though it passed already a decade after the advocation of this model (Hashiguchi and Ueno, 1977). Then, a stress–strain curve with an open hysteresis loop is predicted for the partial unloading–reloading cycle of stress as was criticized by Mróz and Zienkiewicz (1984), and also the Masing effect (Masing, 1926) cannot be described. It would be the reason why this model has hardly been applied to metals which undergo an elastic deformation in a wide range of stress compared with geomaterials, whereas its basic concept seems available to a wide class of elastoplastic materials. Thus, the author (Hashiguchi, 1980b, 1985b) has tried to extend it so that the center of similarity translates with a plastic deformation, and Dafalias (1981) has tried it for the special case limited to the uniaxial loading behavior of metal.

In this paper, a mathematically exact formulation of the subloading surface model is brought to completion by deriving a translation rule of the center of similarity, avoiding a singularity in the field of plastic moduli, and a consistency condition for the subloading surface and by examining the physical meaning of the loading criterion in terms of a strain rate and the associated flow rule for materials with an anisotropic hardening/softening and without an elastic domain. Its capability for prediction of hysteresis behavior including the Masing effect, a closed hysteresis loop and a mechanical ratchetting effect, which are the fundamental properties of cyclic loading behavior in the subyield state, is shown concisely by the analyses of benchmark problems in uniaxial loading. Further, this model is applied to metals by determining material functions explicitly and is compared with test data of the torsional cyclic loading behavior of stainless steel. Finally, mathematically inevitable shortcomings of the other well-known models, i.e. the multi, the infinite, the two and the single surface models are discussed, comparing with the present model.

2. BASIC CONSTITUTIVE EQUATIONS FOR THE NORMAL-YIELD STATE

Constitutive equations for the normal-yield state in which a current stress lies on the normal-yield surface are formulated below, which belong to the framework of conventional theory, and these will be extended to the subloading surface model in the subsequent sections. While some of these equations were described in the previous paper (Hashiguchi, 1985a), they are repeated here since they are necessary for the formulation and explanation of the subloading surface model.

First, assume that the normal-yield surface is described by the following equation:

$$f(\hat{\sigma}) - F(H) = 0 \tag{1}$$

setting

$$\hat{\sigma} \equiv \sigma - \hat{\alpha}. \tag{2}$$

The second-order tensor σ is a stress, and the scalar H and the second-order tensor $\hat{\alpha}$ are internal state variables for describing the expansion/contraction and the translation, respectively, of the surface. Let $f - F < 0$ hold in the interior of the yield surface. For simplicity, one assumes that the surface described by eqn (1) expands/contracts retaining a geometrical similarity in a stress space. Therefore, the function f is to be a homogeneous function which satisfies the relations $f(ax_i) = a^n f(x_i)$ and $\Sigma_i \partial f/\partial x_i \cdot x_i = \mathrm{n} f$ for any real a and variables x_i, where n is the degree of homogeneity of the function f.

Let $\dot{H}$, where a superposed dot designates a material-time derivative, be a function of plastic strain rate $\dot{\varepsilon}^p$ (homogeneous of degree one by dimensional invariance of time) and some plastic internal state variables describing a history of plastic deformation.

Further, let $\dot{\hat{\alpha}}$ be given as

$$\dot{\hat{\alpha}} = \dot{A}\frac{\hat{\sigma}}{\|\hat{\sigma}\|} - \dot{B}\hat{\alpha} \tag{3}$$

where $\dot{A}$ and $\dot{B}$ are functions of $\dot{\varepsilon}^p$ in homogeneity of degree one and some plastic internal state variables, and the notation $\|\ \|$ represents a norm (magnitude). (It can be set that $\dot{A} = 0$ and $\dot{B} = -\dot{F}/F$, resulting in $\dot{\hat{\alpha}} = -\dot{F}\mathbf{I}$, for geomaterials.)

By differentiating eqn (1) and substituting the relation

$$\frac{\partial f(\hat{\sigma})}{\partial \hat{\sigma}} = \frac{\mathrm{n}F}{\mathrm{tr}\,(\hat{\mathbf{n}}\hat{\sigma})}\hat{\mathbf{n}} \quad \text{when } f(\hat{\sigma}) = F \tag{4}$$

$$\hat{\mathbf{n}} \equiv \frac{\partial f(\hat{\sigma})}{\partial \hat{\sigma}} \bigg/ \left\| \frac{\partial f(\hat{\sigma})}{\partial \hat{\sigma}} \right\| \tag{5}$$

which results from eqn (1), noting the homogeneity of the function f, one has the consistency condition

$$\mathrm{tr}\left\{\hat{\mathbf{n}}\left(\dot{\boldsymbol{\sigma}}-\frac{\dot{F}}{\mathrm{n}F}\boldsymbol{\sigma}\right)\right\} = 0. \tag{6}$$

Here, assume that the associated flow rule holds for the normal-yield state:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\hat{\mathbf{n}} \quad (\dot{\lambda} > 0), \tag{7}$$

where $\dot{\lambda}$ is a proportionality factor.

By substituting eqn (7) into eqn (6) $\dot{\lambda}$ is given as follows:

$$\dot{\lambda} = \frac{\mathrm{tr}\,(\hat{\mathbf{n}}\dot{\boldsymbol{\sigma}})}{\hat{D}} \tag{8}$$

where

$$\hat{D} \equiv \mathrm{tr}\left\{\hat{\mathbf{n}}\left(\frac{F'}{\mathrm{n}F}\hat{h}\boldsymbol{\sigma}+\hat{\mathbf{a}}\right)\right\} \tag{9}$$

$$F' \equiv \mathrm{d}F/\mathrm{d}H. \tag{10}$$

Since $\dot{H}$ and $\dot{\boldsymbol{\alpha}}$ involve $\dot{\boldsymbol{\varepsilon}}^p$ in homogeneity of degree one, one can write

$$\dot{H} = \dot{\lambda}\hat{h}, \tag{11}$$

$$\dot{\boldsymbol{\alpha}} = \dot{\lambda}\hat{\mathbf{a}}. \tag{12}$$

$\hat{h}$ and $\hat{\mathbf{a}}$ are scalar and second-order tensor functions of stress and some plastic internal state variables. $\hat{D}$ is called a plastic (or hardening) modulus in conformity with the similarity to the elastic modulus in a uniaxial loading state.

Let an elastic strain rate be given as

$$\dot{\boldsymbol{\varepsilon}}^e = \mathbf{E}^{-1}\dot{\boldsymbol{\sigma}} \quad \text{or } \dot{\boldsymbol{\sigma}} = \mathbf{E}\dot{\boldsymbol{\varepsilon}}^e \tag{13}$$

where $\mathbf{E}$ (fourth-order tensor) is the elastic modulus.

Substituting eqn (7) with eqn (8) and eqn (13) into the equation

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p, \tag{14}$$

one obtains

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{E}^{-1}\dot{\boldsymbol{\sigma}} + \frac{\mathrm{tr}\,(\hat{\mathbf{n}}\dot{\boldsymbol{\sigma}})}{\hat{D}}\hat{\mathbf{n}} \tag{15}$$

or inversely

$$\dot{\boldsymbol{\sigma}} = \mathbf{E}\left\{\dot{\boldsymbol{\varepsilon}} - \frac{\mathrm{tr}\,(\hat{\mathbf{n}}\mathbf{E}\dot{\boldsymbol{\varepsilon}})}{\hat{D}+\mathrm{tr}\,(\hat{\mathbf{n}}\mathbf{E}\hat{\mathbf{n}})}\hat{\mathbf{n}}\right\}. \tag{16}$$

The constitutive equation (15) or (16) by itself falls within the framework of the conventional theory of elastoplasticity in which the interior of the normal-yield surface is assumed to be an elastic domain. Therefore

(1) the discontinuous stress rate–strain rate relation is predicted, which changes abruptly at the moment when the stress reaches the normal-yield surface;

(2) the loading criterion requires the judgement whether the current stress lies on the normal-yield surface or not;

(3) obviously, the hysteresis loop for the partial unloading–reloading, Masing effect and the mechanical ratchetting phenomenon cannot be described. It is inapplicable to the cyclic loading behavior in the subyield state.

3. FUNDAMENTAL ASSUMPTIONS AND THEIR PHYSICAL INTERPRETATIONS

As was described at the end of Section 2, the elastoplastic constitutive model in which the interior of the normal-yield surface is assumed to be an elastic domain has fundamental limitations. In the following, let the models in this structure be called "conventional (elastoplastic constitutive) models", and let the extended models (e.g. the multi, the infinite, the two and the subloading surface models) to the subyield state be called "unconventional models" in accordance with Drucker (1988). Prior to extending the equations in Section 2 to the subloading surface model as an unconventional model, the fundamental assumptions for new formulations and their physical interpretations are given in this section.

3.1. *Assumptions*

The following assumptions are incorporated into the subloading surface model which will be formulated so as to overcome the aforementioned limitations in the conventional model.

[I] *The surface, called a "subloading surface", exists, which expands/contracts within the normal-yield surface, passing always through a current stress point not only in a loading (elastoplastic) process but also in an unloading (elastic) process.*

[II] *The subloading surface is similar to the normal-yield surface, and these surfaces lie in positions of similarity, preserving the same orientation without relative rotation.*

By the assumption [II] a center of similarity (or similarity-center) exists for the specified configuration of the normal-yield and the subloading surfaces. Let the position vector of similarity-center be denoted by S. Besides, in view of the assumption [I], the similarity-center must lie inside the normal-yield surface.

The similarity-center of two figures is characterized by the fact that the straight lines issuing from it intersect with the corresponding (or conjugate) points on these figures in a constant ratio of distances from the center, provided these figures are not only similar but also are located in positions of similarity. In the case of two surfaces whose geometrical centers are specified, the above-mentioned straight lines intersect with these surfaces and with their centers in a constant ratio of distances from the similarity-center. Needless to say, the geometrical centres of these surfaces are different from each other and also are different from the similarity-center in general.

[II′] *The similarity-center does not lie on the normal-yield surface.*

In the state that the similarity-center lies on the normal-yield surface, the subloading and the normal-yield surfaces contact with each other in their different sizes. If the stress coincides with similarity-center in this situation, the aforementioned ratio of the distances becomes indefinite so that a subloading surface is not determined uniquely. Thus, the contact point, i.e. the similarity-center, becomes the singular point of a field of elastic–plastic moduli which causes a discontinuity of stress rate–strain rate relation, while the subloading surface plays as a loading surface as will be described later as to the assumption [IV]. This is physically inadmissible in general, although it has been widely assumed following the first proposition of this model (Hashiguchi and Ueno, 1977) for soils in such a way that the similarity-center is fixed in the origin of stress space and the normal-yield surface passes through it. Eventually, this assumption is required to guarantee that the subloading surface is always uniquely determined and thus a continuous stress rate–strain rate is always described for the non-zero strain rate. (The indifferentiability of a stress rate with respect to a strain rate in the neighborhood of the null strain rate is the fundamental property of the irreversible deformation although Truesdell (1955) excluded it in the hypoelastic equation.)

[III] *The similarity-center moves (more excatly, can move only) during a loading (elastoplastic) process but does not move during an unloading (elastic) process.*

In view of this assumption the similarity-center can be regarded as a plastic internal state variable as well as F and $\hat{\alpha}$ in the conventional model. On the other hand, the geometrical center of the subloading surface, denoted by $\bar{\alpha}$, is not a plastic internal variable as it evolves even during an unloading (elastic) process in accordance with the assumption [I]. Whereas, $\bar{\alpha}$ is determined from the geometrical relations of σ, F, $\hat{\alpha}$ and S since the subloading surface is similar and is located in a position of similarity to the normal-yield surface.

Now, one introduces the ratio of the size of the subloading surface to that of the normal-yield surface. Let it be called a "NS–surface size ratio" (abbreviated as "NSR") and let it be denoted by R. Needless to say, NSR ranges from zero to unity. Hence,

[IV] *NSR increases and approaches unity when a plastic deformation occurs. Inversely, a plastic deformation occurs when NSR increases.*

By this assumption NSR decreases or does not change when a purely elastic deformation occurs, and inversely a purely elastic deformation only can occur when NSR decreases or does not change. Thus, the subloading surface plays a role of loading surface. This is a physical background of the term "subloading surface". Besides, it is not required to judge whether a stress lies on the loading surface or not in a loading criterion since a stress always lies on it by the assumption [I], while the judgement whether a stress lies on the yield surface or not is required in conventional models.

[IV′] *A plastic deformation generated in the null NSR state is infinitesimal.*

Now, note that the null NSR state ($R = 0$: $\sigma = \bar{\alpha} = \mathrm{S}$) is the minimum state of NSR since $R \geqslant 0$. Then, by the assumption [IV], only a purely elastic deformation for $\dot{R} < 0$ can occur to reach the null NSR state, and after that an elastoplastic deformation for $\dot{R} > 0$ occurs. Now, if a plastic deformation occurs finitely in the null NSR state, a stress rate–strain rate relation becomes discontinuous in this state by the abrupt occurrence of plastic deformation even if a stress path is smooth. In other words, the null NSR state becomes a singular point of the field of elastic–plastic moduli. In order to avoid this physical and mathematical shortcoming, the assumption [IV] is accompanied with the subsidiary assumption [IV′]. Thus, it results that the ratio of the rate of NSR, i.e. $\dot{R}$, to that of the magnitude of plastic strain rate is infinite in the null NSR state. Eventually, a purely elastic deformation occurs substantially in the null NSR state.

Besides, by the assumptions [IV] and [IV′], the state in which a purely elastic deformation occurs for stress rates of any direction is realized in the null NSR state. In other words, an elastic domain exists merely as a point and only in the position of similarity-center.

[V] *When NSR is unity, i.e. in the normal-yield state, a stress rate–strain rate relation is given by the conventional equations described in Section 2.*

This assumption leads to that the subloading surface model formulated later is the extension of the conventional model and thus it does not leap from it. Consequently, all the equations described in Section 2 hold in the normal-yield state.

3.2. *Physical interpretations of assumptions*

In the initial subloading model (Hashiguchi and Ueno, 1977; Hashiguchi, 1978, 1980a) or the bounding surface model with a radial mapping (Dafalias and Herrmann, 1980), the similarity-center is fixed in the origin of stress space or in a certain point within the normal-yield surface. As the simplest case one considers the uniaxial loading behavior of the idealized material with the nonhardening Mises normal-yield surface and with an initial isotropy as shown in Fig. 1 in which σ and ε^p are axial components of σ and ε^p, respectively. By the initial subloading or the bounding surface model with a radial mapping, all the shapes of initial loading, reverse loading and reloading curves are predicted to be the same and only an elastic deformation is predicted in an unloading process since the subloading surface shrinks as a stress decreases. Therefore, the Masing effect is not described and an open hysteresis loop is predicted. These shortcomings are caused by the structure of this model in which the similarity-center is fixed.

On the other hand, let the similarity-center move with a plastic deformation as shown in Fig. 2 in which S and $\bar{\alpha}$ are axial components of the similarity-center S and the center of the subloading surface, $\bar{\boldsymbol{\alpha}}$, respectively. By the premise of initial isotropy, the similarity-center lies at the origin of stress space and the subloading surface is merely a point without a size at the onset of initial loading as shown in Fig. 2(a) and it expands gradually as the stress increases so that a plastic deformation is generated and therefore the similarity-center also moves up following a stress as shown in Fig. 2(b). On the other hand, in the unloading state shown in Fig. 2(c), the subloading surface shrinks gradually and reduces to a point when the stress decreases to the position of the similarity-center so that only an elastic deformation is generated and therefore the similarity-center does not move in this process. But after the stress passed through the position of similarity-center the subloading surface expands again from the point so that a plastic deformation is generated gradually, and therefore the similarity-center moves as shown in Fig. 2(d). In other words, a plastic deformation begins before a stress vanishes so that the Masing rule can be described to some extent. Further, in the reloading process shown in Fig. 2(e), the subloading surface shrinks gradually and reduces to the point when a stress increases to the position of the similarity center so that only an elastic deformation is generated and the similarity-center does not move in this process similarly to the initial stage of unloading mentioned above. Subsequently, the subloading surface expands so that a plastic deformation proceeds and the similarity-center moves up following the stress as shown in Fig. 2(f). A description of closed hysteresis loop is attained in this manner, whereas, the reloading after a small unloading in a purely elastic deformation (i.e. the increase of σ prior to its decrease to S in Fig. 2(e)) causes an open hysteresis loop.

Physical meaning of similarity center. The Bauschinger effect means that the yield stress in the reverse loading becomes smaller than that in the initial loading, inducing a plastic deformation. It gives rise to the induced anisotropy of plastic deformation behavior in the normal-yield state. This effect is described concisely by the kinematic hardening in which the center $\hat{\boldsymbol{\alpha}}$ of normal-yield surface moves with a plastic deformation. Here, $\hat{\boldsymbol{\alpha}}$ is regarded to be a geometrical center of elastic domain. On the other hand, the Masing rule is characterized by the fact that a curvature of the reverse loading curve becomes smaller than that of the initial loading curve. Further, a closed hysteresis loop during the unloading–reloading process is caused by a small plastic deformation in the unloading process prior to a purely elastic deformation at the onset of reloading. These phenomena are interpreted to be caused by the fact that the stress state, in which materials deform most elastically, is not fixed in its null state but moves following a current stress during a plastic deformation. As was described as to the assumptions [IV] and [IV′], the similarity-center S expresses this

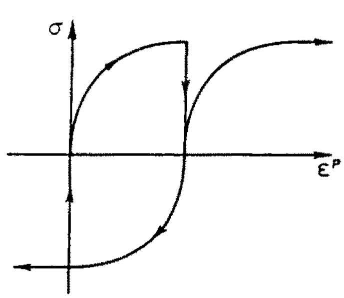

Fig. 1. A schematic diagram of uniaxial loading behavior predicted by the initial subloading surface model (bounding surface model with a radial mapping).

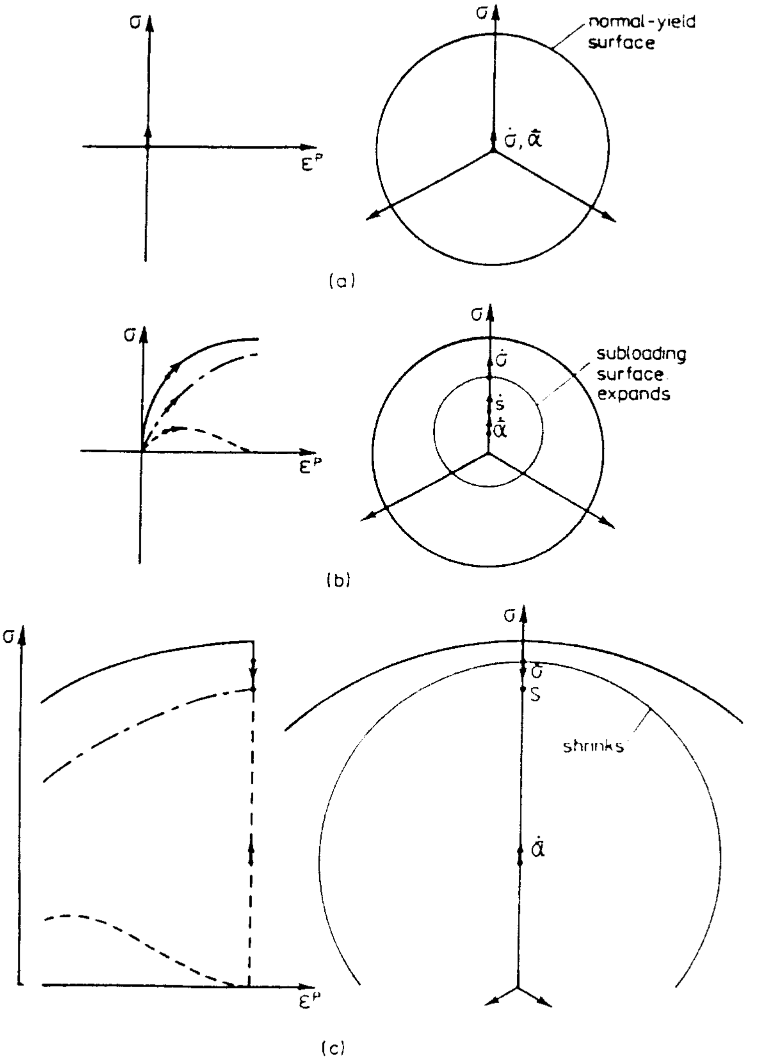

Fig. 2. A schematic diagram of uniaxial loading behavior predicted by the subloading surface model: (a) a beginning state of an initial loading ($\sigma = S = \bar{\alpha} = 0$, $R = 0$); (b) an initial loading process ($\dot{\sigma} > 0$, $\dot{S} > 0$, $\dot{R} > 0$); (c) an unloading process until σ decreases to S ($\dot{\sigma} < 0$, $\dot{S} = 0$, $\dot{R} < 0$: elastic deformation); (d) an unloading reverse loading process after σ passed through S($\dot{\sigma} < 0$, $\dot{S} < 0$, $\dot{R} > 0$); (e) a reloading process until σ increases to S ($\dot{\sigma} > 0$, $\dot{S} = 0$, $\dot{R} < 0$: elastic deformation); (f) a reloading process after σ passed through S ($\dot{\sigma} > 0$, $\dot{S} > 0$, $\dot{R} > 0$). ——— σ, — - — S, - - - - $\bar{\alpha}$.

stress state, called "the most elastic stress", and its movement gives rise to the induced anisotropy which affects a response for a small plastic deformation in the subyield state. Thus, while the center of normal-yield surface, $\boldsymbol{\alpha}$, can be called a "geometrical center of elastic domain" (in the conventional sense) or a "normal-yield back stress" or a "normal-yield kinematic hardening parameter", the similarity-center S can be called "the most elastic stress" or a "subyield back stress" or a "subyield kinematic hardening parameter". Also, F can be called a "size of elastic domain" (in the conventional sense) or an "isotropic hardening parameter".

The conventional isotropic/kinematic hardening model involves only two internal variables, i.e. F and $\boldsymbol{\alpha}$. On the other hand, the extended subloading surface model involves three internal variables, i.e. F, $\boldsymbol{\alpha}$ and S. Then, the formulation of the evolution equation of the similarity-center S is to be the main problem in the extension of the conventional model to the unconventional one.

Physical meaning and role of NSR. It seems plausible to assume that the plastic deformation occurs when a subloading surface expands. During a softening process,

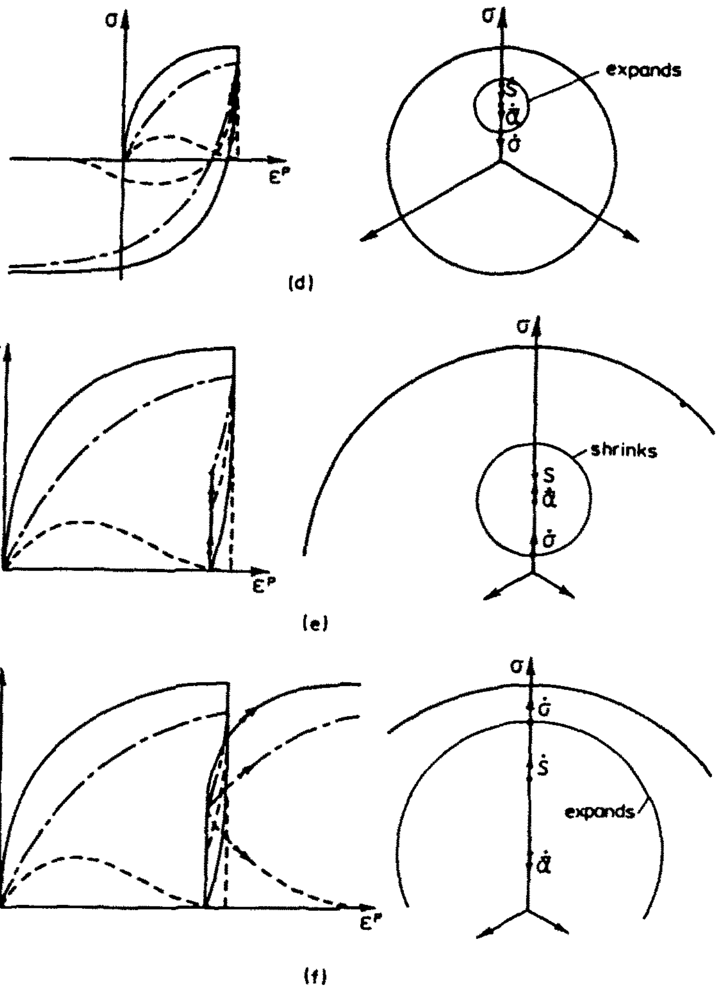

Fig. 2. (continued).

however, the normal-yield surface itself shrinks and thus the subloading surface also shrinks when they approach closely one to the other. Therefore, it cannot be assumed in general that a plastic deformation does not occur when the subloading surface shrinks. This is the physical background of the assumption [IV] which is described by the NSR (not by the expansion/contraction of the subloading surface itself). Besides, this assumption should be incorporated into the formulation of constitutive equation. If not, it is not guaranteed that a stress approaches the normal-yield surface even when a plastic deformation proceeds infinitely. Its incorporation will be done in a formulation of the "extended consistency condition" for this model in which a stress does not lie generally on the normal-yield surface, while in all other models including the multi, the infinite, the two and the bounding surface models, their plastic strain rate equations have been assumed *a priori* by using some *interpolation rule* for plastic moduli between the elastic and the normal-yield states.

As was described as to the assumptions [IV] and [IV′], this model involves an elastic domain as a point. However, almost purely elastic behavior can be described in the subyield state by selecting the high plastic modulus as a function of R. In other words, it can be reduced to the conventional model.

4. FORMULATION OF SUBLOADING SURFACE MODEL

Based on the assumptions described in Section 3, let the subloading surface model be formulated in this section.

The subloading surface is described by the assumption [II] on similarity of the subloading surface to the normal-yield surface as

$$f(\bar{\boldsymbol{\sigma}}) = R^{n}F \tag{17}$$

in setting

$$\bar{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \bar{\boldsymbol{\alpha}} \tag{18}$$

where the function $f(\bar{\boldsymbol{\sigma}})$ has the same form as the homogeneous function $f(\hat{\boldsymbol{\sigma}})$ in eqn (1). R is described by current values of $\boldsymbol{\sigma}$, $\bar{\boldsymbol{\alpha}}$ and F as

$$R \equiv \left\{\frac{f(\bar{\boldsymbol{\sigma}})}{F}\right\}^{1/n} \quad (0 \leqslant R \leqslant 1). \tag{19}$$

Also, by the assumption [II] on similarity, the following geometrical relations hold (see Fig. 3)

$$\tilde{\boldsymbol{\sigma}} = R\tilde{\boldsymbol{\sigma}}_{y} \tag{20}$$

$$\bar{\mathrm{S}} = R\hat{\mathrm{S}} \tag{21}$$

$$\bar{\boldsymbol{\sigma}} = R\hat{\boldsymbol{\sigma}}_{y} \tag{22}$$

$$\bar{\mathbf{n}} = \hat{\mathbf{n}}_{y} \tag{23}$$

where

$$\tilde{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \mathrm{S} \tag{24}$$

$$\tilde{\boldsymbol{\sigma}}_{y} = \boldsymbol{\sigma}_{y} - \mathrm{S} \tag{25}$$

$$\bar{\mathrm{S}} \equiv \mathrm{S} - \bar{\boldsymbol{\alpha}} \tag{26}$$

$$\hat{\mathrm{S}} \equiv \mathrm{S} - \hat{\boldsymbol{\alpha}} \tag{27}$$

$$\hat{\boldsymbol{\sigma}}_{y} \equiv \boldsymbol{\sigma}_{y} - \hat{\boldsymbol{\alpha}} \tag{28}$$

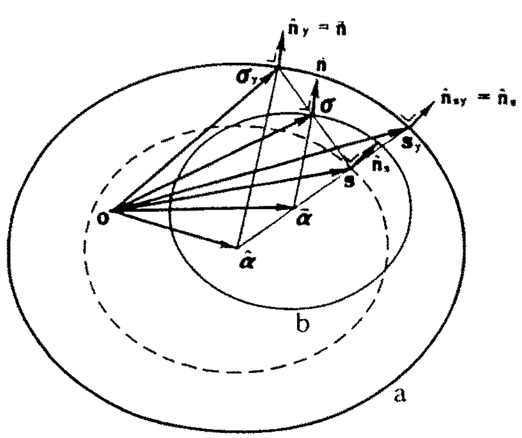

Fig. 3. Configurations of $\boldsymbol{\sigma}$, S, $\hat{\boldsymbol{\alpha}}$ and $\bar{\boldsymbol{\alpha}}$. (a) Normal-yield surface; (b) subloading surface.

$$\bar{\mathbf{n}} \equiv \frac{\partial f(\bar{\sigma})}{\partial \bar{\sigma}} \bigg/ \left\| \frac{\partial f(\sigma)}{\partial \bar{\sigma}} \right\| \tag{29}$$

$$\hat{\mathbf{n}}_y \equiv \frac{\partial f(\hat{\sigma}_y)}{\partial \hat{\sigma}_y} \bigg/ \left\| \frac{\partial f(\hat{\sigma}_y)}{\partial \hat{\sigma}_y} \right\|. \tag{30}$$

σ_y denotes the conjugate stress on the normal-yield surface for the current stress σ on the subloading surface, while the outward normals at these stresses on the surfaces have the same direction.

In the above, there appear the variables σ, F, α, S, R, $\bar{\alpha}$, and σ_y. There exist four independent variables among them. Now, let the expressions of R and $\bar{\alpha}$ by the basic variables, i.e. the current stress σ and the plastic internal state variables F, α and S be explained below, which is required in calculation of stress rate–strain rate.

Substituting the relation

$$\bar{\sigma} = \tilde{\sigma} + R\hat{\mathrm{S}} \tag{31}$$

which is obtained from eqn (21), into eqn (17), one has

$$f(\tilde{\sigma} + R\hat{\mathrm{S}}) = R^n F \tag{32}$$

from which one can determine R substituting the current stress σ and internal variables F, α and S. Further, substituting α and S and the already determined R into the equation

$$\bar{\alpha} = \mathrm{S} - R\hat{\mathrm{S}} \tag{33}$$

which is obtained from eqn (31), $\bar{\alpha}$ is determined.

Now, let the extended evolution equation of α to the subyield state be formulated. In accordance with the assumptions [II] and [V], assume that the translation rule (3) of the normal-yield surface holds even in the subyield state, regarding σ in eqn (3) as a conjugate stress σ_y. Hence, noting the relation (22), eqn (3) becomes

$$\dot{\alpha} = \dot{A}\frac{\bar{\sigma}}{\|\bar{\sigma}\|} - \dot{B}\hat{\alpha}. \tag{34}$$

Next, consider the evolution equation of the similarity-center. It must hold by the assumptions [I], [II] and [II′] that

$$f(\hat{\mathrm{S}}) \leqslant \chi^n F \tag{35}$$

or

$$R_s \leqslant \chi \tag{36}$$

where

$$R_s \equiv \left\{\frac{f(\hat{\mathrm{S}})}{F}\right\}^{1/n} \quad (0 \leqslant R_s \leqslant 1). \tag{37}$$

$\chi (0 \leqslant \chi < 1)$ is a material constant. The surface described by $f(\hat{\mathrm{S}}) = R_s^n F$ is depicted by the dashed line in Fig. 3.

Equation (35) is rewritten as

$$\mathrm{tr}\left\{\hat{\mathbf{n}}_s\left(\dot{\hat{\mathrm{S}}}-\frac{\dot{F}}{\mathrm{n}F}\hat{\mathrm{S}}\right)\right\}\leqslant 0 \quad \text{when } R_s=\chi \tag{38}$$

in a differential form, using the relation

$$\frac{\partial f(\hat{\mathrm{S}})}{\partial\hat{\mathrm{S}}}=\frac{\mathrm{n}\chi^{\mathrm{n}}F}{\mathrm{tr}\,(\hat{\mathbf{n}}_s\hat{\mathrm{S}})}\hat{\mathbf{n}}_s \quad \text{when } R_s=\chi \tag{39}$$

where

$$\hat{\mathbf{n}}_s\equiv\frac{\partial f(\hat{\mathrm{S}})}{\partial\hat{\mathrm{S}}}\bigg/\left\|\frac{\partial f(\hat{\mathrm{S}})}{\partial\hat{\mathrm{S}}}\right\|. \tag{40}$$

Equations (35) or (36) and (38) will be called an "enclosing condition of similarity-center".

In order to satisfy eqn (38), referring to Fig. 3, assume that

$$\dot{\hat{\mathrm{S}}}-\frac{\dot{F}}{\mathrm{n}F}\hat{\mathrm{S}}=C\,\|\dot{\boldsymbol{\varepsilon}}^p\|\,(\boldsymbol{\sigma}_y-\mathrm{S}_y) \quad \text{when } R_s=\chi \tag{41}$$

where S_y designates the intersecting point of the normal-yield surface and the straight line issuing from the point $\hat{\boldsymbol{\alpha}}$ and passing through the point S in the stress space, i.e.

$$\mathrm{S}_y=\hat{\boldsymbol{\alpha}}+\frac{\hat{\mathrm{S}}}{R_s}. \tag{42}$$

$C(\geqslant 0)$ is a material constant which controls the rate of translation of the similarity-center. On the other hand, for $R_s=0$, i.e. $\mathrm{S}=\hat{\boldsymbol{\alpha}}$, one assumes that

$$\dot{\hat{\mathrm{S}}}-\frac{\dot{F}}{\mathrm{n}F}\hat{\mathrm{S}}=C\,\|\dot{\boldsymbol{\varepsilon}}^p\|\,(\boldsymbol{\sigma}_y-\hat{\boldsymbol{\alpha}}) \quad \text{when } R_s=0. \tag{43}$$

In Fig. 3, $\hat{\mathbf{n}}_{sy}$ is the outward normal of the normal-yield surface at the point S_y, i.e.

$$\hat{\mathbf{n}}_{sy}\equiv\frac{\partial f(\hat{\mathrm{S}}_y)}{\partial\mathrm{S}_y}\bigg/\left\|\frac{\partial f(\hat{\mathrm{S}}_y)}{\partial\hat{\mathrm{S}}_y}\right\| \tag{44}$$

setting

$$\hat{\mathrm{S}}_y\equiv\mathrm{S}_y-\hat{\boldsymbol{\alpha}}. \tag{45}$$

For eqns (41) and (43) to be satisfied, one assumes the following linear equation of R_s as the simplest one

$$\dot{\hat{\mathrm{S}}}-\frac{\dot{F}}{\mathrm{n}F}\hat{\mathrm{S}}=C\,\|\dot{\boldsymbol{\varepsilon}}^p\|\left\{\boldsymbol{\sigma}_y-\hat{\boldsymbol{\alpha}}+\frac{R_s}{\chi}(\hat{\boldsymbol{\alpha}}-\mathrm{S}_y)\right\} \tag{46}$$

which is rewritten as

$$\dot{\hat{\mathrm{S}}}-\frac{\dot{F}}{\mathrm{n}F}\hat{\mathrm{S}}=C\,\|\dot{\boldsymbol{\varepsilon}}^p\|\left(\frac{\tilde{\boldsymbol{\sigma}}}{R}-\frac{\hat{\mathrm{S}}}{\chi}\right), \tag{47}$$

noting eqns (22) and (42).

Eventually, the translation rule of S is given from eqn (47) as follows:

$$\dot{\mathbf{S}} = \dot{\boldsymbol{\alpha}} + \frac{\dot{F}}{\mathrm{n}F}\hat{\mathbf{S}} + C\|\dot{\boldsymbol{\varepsilon}}^p\|\left(\frac{\tilde{\boldsymbol{\sigma}}}{R} - \frac{\hat{\mathbf{S}}}{\chi}\right) \tag{48}$$

which reduces to

$$\dot{\mathbf{S}} = \frac{C}{R}\|\dot{\boldsymbol{\varepsilon}}^p\|\tilde{\boldsymbol{\sigma}} = C\|\dot{\boldsymbol{\varepsilon}}^p\|\tilde{\boldsymbol{\sigma}}_y \quad \text{when } \dot{F} = 0,\ \ \dot{\boldsymbol{\alpha}} = \mathbf{0},\ \ \chi = 1. \tag{49}$$

The evolution equation of $\dot{\boldsymbol{\alpha}}$ given by eqn (3) conforms to Ziegler's (1959) modification of Prager's (1956) kinematic hardening rule due to a mathematical convenience that the components of $\dot{\boldsymbol{\alpha}}$ in the directions of null stress condition vanish throughout a deformation for initially isotropic materials. However, it does not differ from Prager's rule in the case of metals with von Mises yield surface. Equation (34) is the extension of eqn (3) to the subyield state. Further, the evolution equation of the similarity-center given by eqn (48) also involves this mathematical convenience eventually. Then, all the components of $\dot{\boldsymbol{\alpha}}$, S and $\bar{\boldsymbol{\alpha}}$ [see eqn (33)] in the directions of null stress condition vanish consistently throughout a deformation, while a more due consideration is required to clarify whether it has a physically inevitable reason too.

Next, one formulates a consistency condition for this model in which a current stress does not lie on the normal-yield surface in general.

Differentiating eqn (32) and noting the relation

$$\frac{\partial f(\bar{\boldsymbol{\sigma}})}{\partial \bar{\boldsymbol{\sigma}}} = \frac{\mathrm{n}R^{\mathrm{n}}F}{\mathrm{tr}\,(\bar{\mathbf{n}}\bar{\boldsymbol{\sigma}})}\bar{\mathbf{n}}, \tag{50}$$

one has

$$\mathrm{tr}\left\{\bar{\mathbf{n}}\left(\dot{\boldsymbol{\sigma}} + R\dot{\mathbf{S}} - \frac{\dot{F}}{\mathrm{n}F}\bar{\boldsymbol{\sigma}} - \frac{\dot{R}}{R}\tilde{\boldsymbol{\sigma}}\right)\right\} = 0. \tag{51}$$

In order to obtain from eqn (51) a consistency condition by which a plastic strain rate will be formulated, let an evolution equation of R, i.e. $\dot{R}$, be assumed. In accordance with the assumptions [IV] and [IV′], one introduces the equation

$$\dot{R} = U\|\dot{\boldsymbol{\varepsilon}}^p\| \quad \text{for } \dot{\boldsymbol{\varepsilon}}^p \neq \mathbf{0} \tag{52}$$

where U is a monotonically decreasing function with respect to R satisfying the conditions

$$\begin{aligned} U &= +\infty \quad \text{for } R = 0 \\ U &= 0 \quad \text{for } R = 1. \end{aligned} \tag{53}$$

The Masing rule and a closed hysteresis loop are described to some extent by the movement of the most elastic stress S. However, note that if U is a function of R alone, eqn (52) results in $R - R_0 = f(\bar{\varepsilon}^p - \bar{\varepsilon}^p_0)$ for $R = R_0$: $\bar{\varepsilon}^p = \bar{\varepsilon}^p_0$ where $\bar{\varepsilon}^p$ is the accumulated plastic strain, i.e. $\bar{\varepsilon}^p = \int \|\dot{\boldsymbol{\varepsilon}}^p\|\,\mathrm{d}t$ (t, time). Therefore, the accumulated plastic strain $\bar{\varepsilon}^p$ generated until R reaches a certain value from a certain state (R_0, $\bar{\varepsilon}^p_0$) in a loading (elasto-plastic) process is the same independent of an initial loading, reverse loadings and reloadings after unloadings of various magnitudes. This would not be realistic as known from the fact, for example, that the plastic strain generated during the reloading process after a small unloading is to be far smaller than that during the initial loading process. Then, assume that the function U depends not only on R but also on a distance from the current stress $\boldsymbol{\sigma}$ to the conjugate stress $\boldsymbol{\sigma}_y$. As a measure of this distance, one introduces the non-dimensional parameter

$$R_y \equiv \left\{\frac{f(\sigma_y-\sigma)}{F}\right\}^{1/n} \tag{54}$$

which is rewritten by eqn (20) as

$$R_y = \left(\frac{1}{R}-1\right)\tilde{R} \tag{55}$$

where

$$\tilde{R} \equiv \left\{\frac{f(\tilde{\sigma})}{F}\right\}^{1/n}. \tag{56}$$

Thus, the function U is given as

$$U = \hat{U}(R, R_y) \quad \text{or } U = \hat{U}(R, \tilde{R}). \tag{57}$$

Obviously U must be a monotonically decreasing function with respect to R_y or $\tilde{R}$ too. Examples of the function U are

$$U = u_1(1-R^m)/\tilde{R} \tag{58}$$

$$U = -u_2 \ln R/\tilde{R} \tag{59}$$

where u_1, u_2 and m are material constants.

Substituting eqns (48) and (52) into eqn (51), one obtains the consistency condition for this extended model:

$$\mathrm{tr}\left[\bar{\mathbf{n}}\left(\dot{\sigma}-\frac{\dot{F}}{nF}\sigma-\left\{C(1-R)\left(\frac{\bar{\sigma}}{R}-\frac{\hat{\mathrm{S}}}{\chi}\right)+\frac{U}{R}\bar{\sigma}\right\}\|\dot{\varepsilon}^p\|\right)\right]=0. \tag{60}$$

Here, assume that the associated flow rule holds also for the subloading surface:

$$\dot{\varepsilon}^p = \dot{\bar{\lambda}}\bar{\mathbf{n}}. \tag{61}$$

By substituting eqn (61) into (60), the proportionality factor $\dot{\bar{\lambda}}$ is given as

$$\dot{\bar{\lambda}} = \frac{\mathrm{tr}\,(\bar{\mathbf{n}}\dot{\sigma})}{\bar{D}}, \tag{62}$$

where

$$\bar{D} \equiv \mathrm{tr}\left[\bar{\mathbf{n}}\left\{\frac{F'}{nF}\bar{h}\sigma+\bar{\mathbf{a}}+C(1-R)\left(\frac{\bar{\sigma}}{R}-\frac{\hat{\mathrm{S}}}{\chi}\right)+\frac{U}{R}\bar{\sigma}\right\}\right]. \tag{63}$$

Since $\dot{H}$ and $\dot{\alpha}$ involve $\dot{\varepsilon}^p$ in homogeneity of degree one, one can write them as

$$\dot{H} = \dot{\bar{\lambda}}\bar{h}, \tag{64}$$

$$\dot{\alpha} = \dot{\bar{\lambda}}\bar{\mathbf{a}}. \tag{65}$$

$\bar{h}$ and $\bar{a}$ are scalar and second-order tensor functions of stress and some plastic internal variables. The loading criterion and the physical interpretation of the associated flow rule for this extended model are given in the next section.

It should be noted that eqns (61)–(63) lead to $\bar{D} \to \infty$ ($\dot{\varepsilon}^p \to \mathbf{0}$) for $R < 1$ and $\bar{D} = \hat{D}$ for $R = 1$ by selecting $U \to \infty$ for $R < 1$ [$u_1 \to \infty$ or $u_2 \to \infty$ in eqn (58) or (59)]. In other words, this model reduces to the classical constitutive equation by the selection of material parameters, the interior of the normal-yield surface becoming an elastic domain substantially.

For the special case of $\mathbf{S} = \hat{\boldsymbol{\alpha}} = \bar{\boldsymbol{\alpha}}$, the function $\bar{D}$ of eqn (63) reduces to

$$\bar{D} = \mathrm{tr}\left[\bar{\mathbf{n}}\left\{\left(\frac{F'}{\mathrm{n}F}\bar{h} + \frac{U}{R}\right)\bar{\boldsymbol{\sigma}} + \bar{\mathbf{a}}\right\}\right]. \tag{66}$$

Further, for $\mathbf{S} = \hat{\boldsymbol{\alpha}} = \bar{\boldsymbol{\alpha}} = \mathbf{0}$ with $\bar{\mathbf{a}} = \mathbf{0}$ and $C = 0$, the function $\bar{D}$ reduces to

$$\bar{D} = \left(\frac{F'}{\mathrm{n}F}\bar{h} + \frac{U}{R}\right)\mathrm{tr}\,(\bar{\mathbf{n}}\boldsymbol{\sigma}). \tag{67}$$

In the above, the plastic modulus $\bar{D}$ was derived logically by formulating the consistency condition on the premise $\dot{R} > 0$ for $\dot{\varepsilon}^p \neq \mathbf{0}$, while the plastic modulus has been assumed *a priori* in the past formulations by interpolation rules (Hashiguchi and Ueno, 1977; Hashiguchi, 1978, 1979, 1980a; Dafalias and Herrmann, 1980; Dafalias, 1986; Zienkiewicz and Mróz, 1984, etc. for $\mathbf{S} = \hat{\boldsymbol{\alpha}} = \bar{\boldsymbol{\alpha}} = \mathbf{0}$ and Hashiguchi, 1980b, 1985b for $\dot{\mathbf{S}} \neq \mathbf{0}$).

Combining elastic and plastic strain rate equations (13) and (61) with eqn (62), one obtains

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{E}^{-1}\dot{\boldsymbol{\sigma}} + \frac{\mathrm{tr}\,(\bar{\mathbf{n}}\dot{\boldsymbol{\sigma}})}{\bar{D}}\bar{\mathbf{n}} \tag{68}$$

or inversely

$$\dot{\boldsymbol{\sigma}} = \mathbf{E}\left\{\dot{\boldsymbol{\varepsilon}} - \frac{\mathrm{tr}\,(\bar{\mathbf{n}}\mathbf{E}\dot{\boldsymbol{\varepsilon}})}{\bar{D} + \mathrm{tr}\,(\bar{\mathbf{n}}\mathbf{E}\bar{\mathbf{n}})}\bar{\mathbf{n}}\right\}. \tag{69}$$

5. A LOADING CRITERION AND THE ASSOCIATED FLOW RULE

A loading criterion formulated in terms of a strain rate instead of a stress rate with the outward normal of plastic potential surface and the elastic modulus was induced by Hill (1958), premising on a hardening process. This criterion seems applicable to generalized elastoplastic materials with hardening/softening behavior. Later on, Hill (1967) formulated it on the postulate that a strain rate space is divided by a hyperplane into the two domains which cause a loading and an unloading, respectively. Besides, Mróz and Zienkiewicz (1984) formulated it on the postulate that a strain space is divided by a yield surface into the two domains which cause a loading and an unloading, respectively. It does not belong to the ordinary stress space formulation but falls within the so-called strain space formulation in which the constitutive relation of a stress rate and a strain rate is formulated by a strain (not a stress) and plastic internal variables. Both of them are not straightforward formulations from the postulate on the ordinary loading surface in the stress space. Whereas, the strain space formulation premises at present that the interior of yield surface is an elastic domain and that the yield surface includes a null stress state, because of the decomposition of strain into elastic and plastic components. Thus, the existing strain space formulation is regarded as the untraditional representation or interpretation of the classical elastoplasticity.

The physical interpretations of the associated flow rule were given by Drucker (1951) and by Ilyushin (1961). The former belongs to the stress space formulation but premises the existence of the yield surface enclosing an elastic domain and the latter belongs substantially to the strain space formulation.

The present model does not premise the existence of an elastic domain and also does not use a strain belonging to the stress space formulation. In this section, a physical interpretation of the above-mentioned loading criterion by Hill is given from the postulate on the loading surface in a stress space within the framework of the stress space formulation for the generalized materials without an elastic domain but with hardening/softening behavior, while the formulation of this loading criterion was discussed briefly by the author (Hashiguchi, 1988) on the premise of the associated flow rule. Further, the associated flow rule is derived from this loading criterion and from Ilyushin's hypothesis of a non-negative work done during a strain cycle. In this section let a loading and an unloading mean the processes during which a plastic deformation occurs and does not occur, respectively.

Now, one introduces a loading surface:

$$f(\boldsymbol{\sigma}, H_i) = 0 \quad (i = 1, 2, \ldots, n) \tag{70}$$

where scalar or tensors H_i denote collectively plastic internal variables, and $f < 0$ for the interior of the loading surface.

Differentiation of eqn (70) leads to the consistency condition

$$\mathrm{tr}\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\dot{\boldsymbol{\sigma}}\right) + \sum_{i=1}^{n} \frac{\partial f}{\partial H_i}\dot{H}_i = 0. \tag{71}$$

Here, let it be assumed that a plastic strain rate $\dot{\boldsymbol{\varepsilon}}^p$ is expressed as

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\mathbf{m} \quad (\dot{\lambda} > 0, \quad \|\mathbf{m}\| = 1) \tag{72}$$

where $\dot{\lambda}$ is a proportionality factor determined below, and $\mathbf{m}$ is a normalized second-order tensor which is a function of stress and some plastic internal variables.

$\dot{H}_i$ can be expressed by eqn (72) as

$$\dot{H}_i = \dot{\lambda} h_i \tag{73}$$

where scalars or tensors h_i are functions of stress and some plastic internal variables.

Substituting eqn (73) into eqn (71), we have

$$\dot{\lambda} = \frac{\mathrm{tr}\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\dot{\boldsymbol{\sigma}}\right)}{-\displaystyle\sum_{i=1}^{n} \frac{\partial f}{\partial H_i} h_i} \quad (>0 \quad \text{for } \dot{\boldsymbol{\varepsilon}}^p \neq \mathbf{0}) \tag{74}$$

or

$$\dot{\lambda} = \frac{\mathrm{tr}(\mathbf{n}\dot{\boldsymbol{\sigma}})}{\mathfrak{D}} \quad (>0 \quad \text{for } \dot{\boldsymbol{\varepsilon}}^p \neq \mathbf{0}), \tag{75}$$

where

$$\mathfrak{D} \equiv -\sum_{i=1}^{n} \frac{\partial f}{\partial H_i} h_i \Big/ \left\| \frac{\partial f}{\partial \boldsymbol{\sigma}} \right\|, \tag{76}$$

$$\mathbf{n} \equiv \frac{\partial f}{\partial \boldsymbol{\sigma}} \Big/ \left\| \frac{\partial f}{\partial \boldsymbol{\sigma}} \right\|. \tag{77}$$

Substitutions of eqns (13), (14) and (72) into eqn (75) lead to

$$\dot{\lambda} = \frac{\mathrm{tr}\,\{\mathbf{nE}(\dot{\boldsymbol{\varepsilon}} - \dot{\lambda}\mathbf{m})\}}{\mathfrak{D}} \tag{78}$$

from which one obtains the expression for $\dot{\lambda}$ by the strain rate instead of a stress rate. Let it be denoted as $\dot{\Lambda}$:

$$\dot{\Lambda} = \frac{\mathrm{tr}\,(\mathbf{nE}\dot{\boldsymbol{\varepsilon}})}{\mathfrak{D} + \mathrm{tr}\,(\mathbf{nEm})} \quad (>0 \quad \text{for } \dot{\boldsymbol{\varepsilon}}^p \neq \mathbf{0}) \tag{79}$$

by which eqn (72) is rewritten as

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\Lambda}\mathbf{m}. \tag{80}$$

Now, one derives a loading criterion for the constitutive equation which satisfies the following assumptions.

[VI] *A plastic deformation occurs at least when a stress rate has an outward direction of a loading surface.*

[VII] *A strain rate of any direction can occur in any state, while a stress rate cannot occur in arbitrary direction in general (suppose a perfectly plastic and a softening processes).*

The assumption [VI] is described as

$$\dot{\boldsymbol{\varepsilon}}^p \neq \mathbf{0} \quad \text{when } \mathrm{tr}\,(\mathbf{n}\dot{\boldsymbol{\sigma}}) > 0 \tag{81}$$

which means that the inequality $\mathrm{tr}\,(\mathbf{n}\dot{\boldsymbol{\sigma}}) > 0$ is a *sufficient condition for a loading.*

Thus, it must hold that

$$\mathrm{tr}\,(\mathbf{n}\dot{\boldsymbol{\sigma}}) \leqslant 0 \quad \text{when } \dot{\boldsymbol{\varepsilon}}^p = \mathbf{0} \tag{82}$$

which means that the inequality $\mathrm{tr}\,(\mathbf{n}\dot{\boldsymbol{\sigma}}) \leqslant 0$ is a *necessary condition for an unloading.* This condition is not, however, a sufficient condition for this state. (It holds also in the loading with a softening.) Since it holds that $\dot{\boldsymbol{\sigma}} = \mathbf{E}\dot{\boldsymbol{\varepsilon}}$ in an unloading ($\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e$), the necessary condition (82) for an unloading is rewritten as

$$\mathrm{tr}\,(\mathbf{nE}\dot{\boldsymbol{\varepsilon}}) \leqslant 0 \quad \text{when } \dot{\boldsymbol{\varepsilon}}^p = \mathbf{0} \tag{83}$$

by a strain rate instead of a stress rate. As known from the relation $\mathrm{tr}\,[\mathbf{nE}(-\dot{\boldsymbol{\varepsilon}})] = -\mathrm{tr}\,(\mathbf{nE}\dot{\boldsymbol{\varepsilon}})$, the strain rate space is divided into half spaces by the sign of $\mathrm{tr}\,(\mathbf{nE}\dot{\boldsymbol{\varepsilon}})$.

Now, it can be stated from eqn (79) that

(i) if $\mathfrak{D} + \mathrm{tr}\,(\mathbf{nEm}) < 0$, a loading cannot occur for $\mathrm{tr}\,(\mathbf{nE}\dot{\boldsymbol{\varepsilon}}) > 0$;

(ii) if $\mathfrak{D} + \mathrm{tr}\,(\mathbf{nEm}) = 0$, a loading cannot occur except for the special process $\mathrm{tr}\,(\mathbf{nE}\dot{\boldsymbol{\varepsilon}}) = 0$,

while an unloading also cannot occur in the deformation process $\mathrm{tr}\,(\mathbf{nE}\dot{\boldsymbol{\varepsilon}}) > 0$ by the necessary condition (83). The admissible strain rate and stress rate with the signs of $\mathfrak{D}$ and $\mathrm{tr}\,(\mathbf{nEm})$ under the condition $\dot{\Lambda} > 0$ were examined by Mair and Hueckel (1979) in detail.

Accordingly, a deformation bringing about $\operatorname{tr}(\mathbf{nE}\dot{\boldsymbol{\varepsilon}}) > 0$ cannot occur at all in both a loading and an unloading if $\mathfrak{D} + \operatorname{tr}(\mathbf{nEm}) \leqslant 0$. This contradicts the assumption [VII]. Thus, it must hold that

$$\mathfrak{D} + \operatorname{tr}(\mathbf{nEm}) > 0 \tag{84}$$

in order to satisfy the assumptions [VI] and [VII], and by taking account of eqn (84) into the subsidiary condition $\dot{\Lambda} > 0$, it must hold that

$$\operatorname{tr}(\mathbf{nE}\dot{\boldsymbol{\varepsilon}}) > 0 \quad \text{when } \dot{\boldsymbol{\varepsilon}}^p \neq \mathbf{0} \tag{85}$$

which means that the inequality $\operatorname{tr}(\mathbf{nE}\dot{\boldsymbol{\varepsilon}}) > 0$ is a *necessary condition for a loading*.

The necessary conditions (83) and (85) for an unloading and a loading, respectively, exhibit different ranges of the quantity $\operatorname{tr}(\mathbf{nE}\dot{\boldsymbol{\varepsilon}})$ from each other, while only either of these processes can be taken. Then, it results that they are not only necessary but also sufficient conditions for each process. Eventually, a loading criterion is given as

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}}^p \neq \mathbf{0} &: \operatorname{tr}(\mathbf{nE}\dot{\boldsymbol{\varepsilon}}) > 0, \\ \dot{\boldsymbol{\varepsilon}}^p = \mathbf{0} &: \operatorname{tr}(\mathbf{nE}\dot{\boldsymbol{\varepsilon}}) \leqslant 0 \end{aligned} \tag{86}$$

which was shown by Hill (1958), presupposing $\mathfrak{D} > 0$.

Whereas, $\dot{\lambda}$ in eqn (75) is not applicable to hardening/softening materials, since $\dot{\lambda} > 0$ holds not only in a loading but also in an unloading for the state $\mathfrak{D} < 0$ [for which a softening $\operatorname{tr}(\mathbf{n}\dot{\boldsymbol{\sigma}}) < 0$ proceeds if a loading takes place].

Ilyushin (1961) postulated that the work done during a strain cycle is non-negative, i.e.

$$\oint_{c} \operatorname{tr}(\boldsymbol{\sigma}\, \mathrm{d}\boldsymbol{\varepsilon}) \geqslant 0. \tag{87}$$

For an infinitesimal strain cycle, eqn (87) is written as

$$\operatorname{tr}(\mathrm{d}\boldsymbol{\sigma}^p\, \mathrm{d}\boldsymbol{\varepsilon}) \geqslant 0 \quad \text{or } \operatorname{tr}(\dot{\boldsymbol{\sigma}}^p \dot{\boldsymbol{\varepsilon}}) \geqslant 0 \quad \text{when } \dot{\boldsymbol{\varepsilon}}^p \neq \mathbf{0} \tag{88}$$

where $\dot{\boldsymbol{\sigma}}^p$ is a plastic relaxation stress rate, i.e.

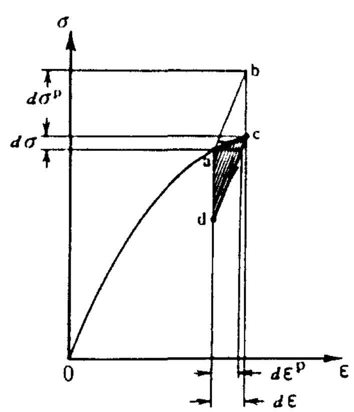

Fig. 4. A non-negative work done during a strain cycle.

$$\dot{\sigma}^p = \mathbf{E}\dot{\varepsilon}^p, \tag{89}$$

referring to Fig. 4 in which the elastic lines ab and cd are regarded as parallel since one considers the infinitesimal strain cycle.

Substituting eqn (80) into eqn (88), one has

$$\mathrm{tr}\,(\mathbf{mE}\dot{\boldsymbol{\varepsilon}}) \geqslant 0 \quad \text{when } \dot{\boldsymbol{\varepsilon}}^p \neq \mathbf{0}, \tag{90}$$

while the strain rate satisfying this inequality occupies a half of strain rate space. In order that eqn (90) is fulfilled for an arbitrary strain rate in the loading process, i.e. eqn $(86)_1$, it must hold that

$$\mathbf{m} = \mathbf{n}. \tag{91}$$

Equations (72) or (80) with eqn (91) is the associated flow rule.

6. QUANTITATIVE DESCRIPTIONS OF BENCHMARK PROBLEMS IN UNIAXIAL LOADING

Let the basic characteristics of the present model be examined by the quantitative descriptions of some benchmark problems in uniaxial loading. In order to do it concisely, one adopts the nonhardening von Mises normal-yield surface. Thus, for a uniaxial loading the normal-yield state is described by

$$|\sigma| = F \tag{92}$$

with

$$\dot{F} = 0, \quad \dot{\alpha} = 0, \quad \mathrm{n} = 1 \tag{93}$$

and eqn (32) is written as

$$|\tilde{\sigma} + RS| = RF \tag{94}$$

from which R is given as

$$R = \frac{\tilde{\sigma}}{\dfrac{\tilde{\sigma}}{|\tilde{\sigma}|}F - S} \tag{95}$$

where $\tilde{\sigma}$ is the axial component of $\tilde{\boldsymbol{\sigma}}$. And $\tilde{\alpha}$ is given from eqns (33) and (95) as

$$\tilde{\alpha} = S(1-R) = S\frac{\dfrac{\tilde{\sigma}}{|\tilde{\sigma}|}F - \sigma}{\dfrac{\tilde{\sigma}}{|\tilde{\sigma}|}F - S}. \tag{96}$$

Let the function U be given as

$$U = u(1 - R^m)/\tilde{R}^\eta, \tag{97}$$

where u, m and η are material constants.

If one adopts a value close to unity for the material constant χ, the evolution equation of the similarity-center is given from eqn (49) by

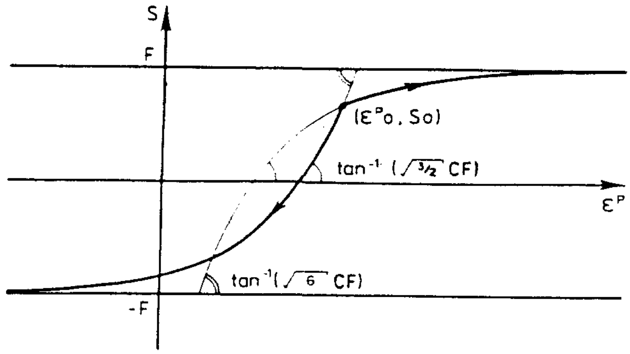

Fig. 5. Relation of S versus ε^p ($\chi = 1$) in uniaxial loading.

$$\dot{S} = C\sqrt{\tfrac{3}{2}}(F \mp S)\dot{\varepsilon}^p \ (\chi \simeq 1). \tag{98}$$

Integration of eqn (98) leads to

$$\frac{S-S_0}{F \mp S_0} = \pm[1-\exp\{C\sqrt{\tfrac{3}{2}}(\varepsilon^p-\varepsilon_0^p)\}] \tag{99}$$

for the initial condition $S = S_0 : \varepsilon^p = \varepsilon_0^p$. In these equations the upper and lower cases stand for $\dot{\varepsilon}^p > 0$ and $\dot{\varepsilon}^p < 0$, respectively. $S - \varepsilon^p$ curves described by eqn (99) have a unique shape as shown in Fig. 5, while the curves for tension and compression are asymmetric to each other.

The axial plastic strain rate is given from eqns (61)–(63) as

$$\dot{\varepsilon}^p = \sqrt{2/3}\,\frac{\dot{\sigma}}{\{C(1-R)+U\}(F \mp S)}. \tag{100}$$

Although the equations for the uniaxial loading are given above in order to exhibit the features of the present model concisely, the calculations were performed by the six-dimensional numerical program based on the exact equations in Section 4 and eqn (97). The calculated results are shown in Fig. 6 in which material constants are selected as

$$F = 100\text{ MPa}, \quad u = 5, \quad m = 5, \quad \eta = 7, \quad C = 700, \quad \chi = 0.99$$

which causes a large plastic deformation compared with usual metals, in order to exhibit a hysteresis and ratchetting behavior clearly.

The initial loading and the unloading-reloading curves are shown in Fig. 6(a) in which a closed loop is observed.

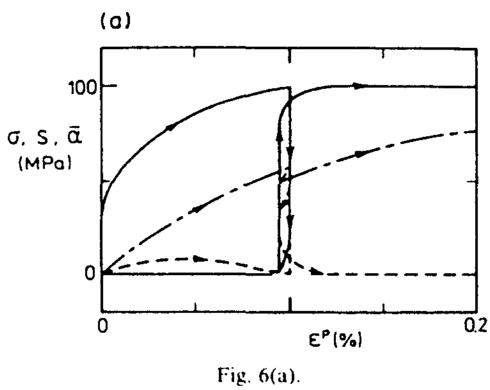

Fig. 6(a).

Fig. 6. Uniaxial loading behavior calculated by the subloading surface model: (a) the initial loading, the unloading and the reloading curves; (b) the hysteresis loop for $|\sigma| \leq 99.5$ MPa; (c) the cyclic loading curve for a large stress amplitude ($\sigma = 0 \sim 90$ MPa); (d) the cyclic loading curve for a small stress amplitude ($\sigma = 50 \sim 90$ MPa). —— σ, — - — S, - - - - - $\bar{\alpha}$.

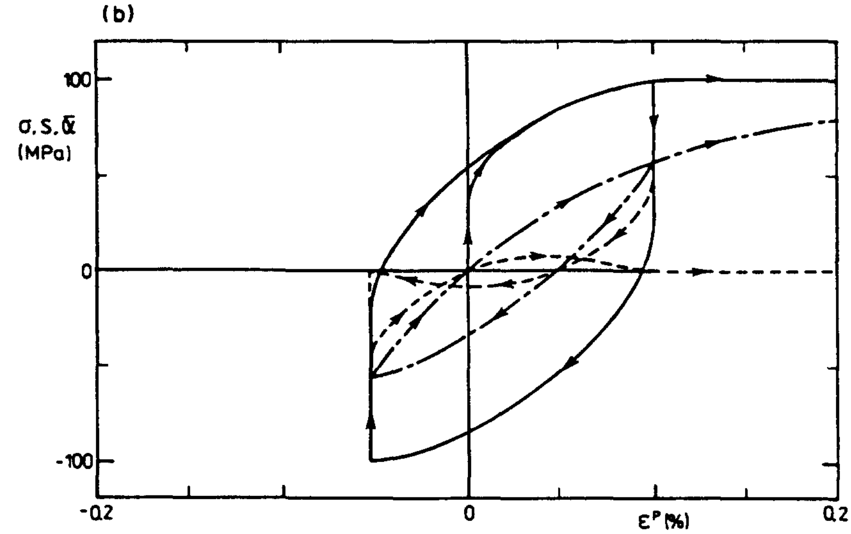

Fig. 6(b).

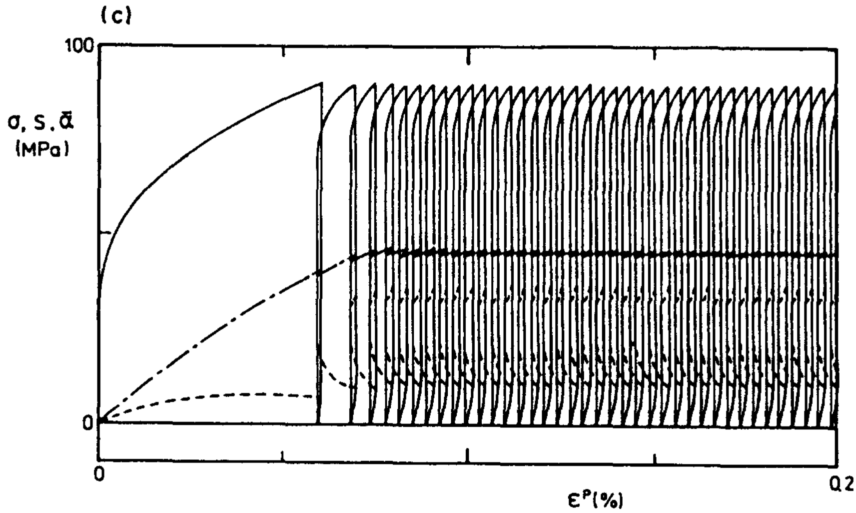

Fig. 6(c).

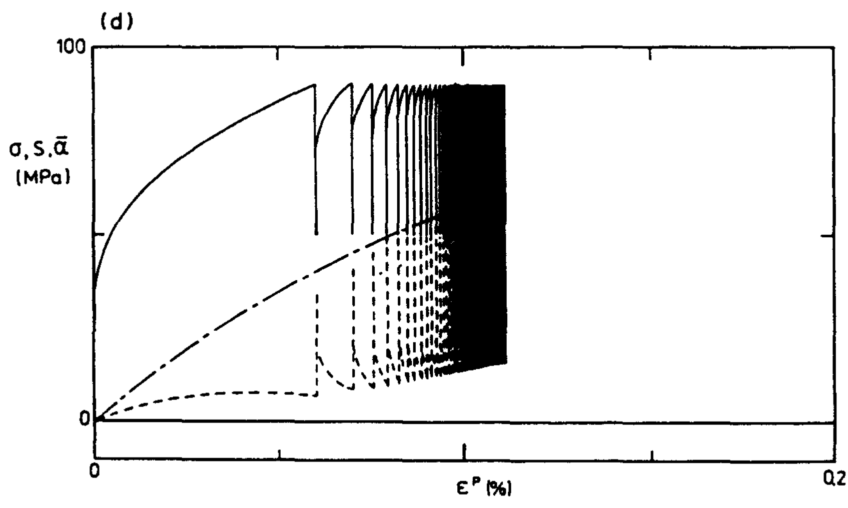

Fig. 6(d).

The hysteresis loop for the range of stress ± 99.5 MPa is shown in Fig. 6(b) in which the Masing rule is expressed to some extent. The Masing rule is described as (Mróz, 1966): When one represents the initial loading curve as $\sigma = f(\varepsilon)$, the reverse loading curve is described as $(\sigma_0 - \sigma)/2 = f((\varepsilon_0 - \varepsilon)/2)$ where σ_0 and ε_0 are values of σ and axial strain ε at the onset of reverse loading. On the other hand, the reverse loading curve in real materials is described as $(\sigma_0 - \sigma)/L = f((\varepsilon_0 - \varepsilon)/L)$ where $L(1 < L < 2)$ can be called a "rounding coefficient", and this phenomenon can be called a "Masing effect" referring to the Baushinger effect. For $L = 1$ the shapes of an initial and a reverse loading curves coincide with each other. On the contrary, for $L = 2$ the stress and the strain generated in a reverse loading process are twice those in an initial loading process, i.e. the Masing rule itself. It is observed in Fig. 6(b) that $L \simeq 1.5$, while L is controllable in the range $1 \leqslant L < 2$ by the selection of the material parameter C: $L = 1$ for $C = 0$, i.e. the initial subloading surface model, and L approaches 2 for $C \gg U$ for which the $\sigma - \varepsilon^p$ curve approaches the $S - \varepsilon^p$ curve shown in Fig. 5.

The cyclic loading behavior for the large stress amplitude 0–90 MPa and the small stress amplitude 50–90 MPa (50 cycles) is shown in Fig. 6(c and d) in which mechanical ratchetting and its shake-down phenomena are observed, while for a small stress amplitude open loops are repeated until S reaches the range of σ. The shake-down is not attained completely and finally the cycle proceeds in a constant interval of plastic strain which is smaller for a smaller stress amplitude. A modification is required to attain a stronger shake-down by making the function U in the evolution equation of R be a function of an accumulated plastic strain $\bar{\varepsilon}^p$ in addition to R and $\tilde{R}$ so that a deformation becomes purely elastic gradually with the increase of $\bar{\varepsilon}^p$.

As shown above by some basic examinations though they are the simple cases, it would be conceivable that the subloading surface model has a basic structure applicable to the prediction of cyclic loading behavior in not only normal-yield but also subyield states.

7. CONSTITUTIVE EQUATIONS OF METALS AND THEIR COMPARISONS WITH AN EXPERIMENT IN SIMPLE TORSIONAL CYCLIC LOADING

The basic formulation of the subloading surface model was given in the preceding sections. Based on it, let the explicit constitutive equations of metals be formulated below in the form applicable to the analysis of cyclic loading behavior.

Let the normal-yield surface be given as

$$f(\hat{\boldsymbol{\sigma}}) = \sqrt{3/2}\,\|\hat{\boldsymbol{\sigma}}'\| \tag{101}$$

$$F(H) = F_0[1 + h_1\{1 - \exp(-h_2 H)\}] \tag{102}$$

$$\dot{\hat{\boldsymbol{\alpha}}} = \|\dot{\boldsymbol{\varepsilon}}^p\| \left(k_1 \frac{\bar{\boldsymbol{\sigma}}}{\|\bar{\boldsymbol{\sigma}}\|} - k_2 \hat{\boldsymbol{\alpha}} \right) \tag{103}$$

$$\dot{H} = \sqrt{2/3}\,\check{R}^{\nu} \|\dot{\boldsymbol{\varepsilon}}^p\| \tag{104}$$

where F_0 is an initial value of F, and h_1, h_2, k_1, k_2 and ν are material constants, and

$$\hat{\boldsymbol{\sigma}}' \equiv \hat{\boldsymbol{\sigma}} - \tfrac{1}{3} \operatorname{tr} \hat{\boldsymbol{\sigma}} \mathbf{I}. \tag{105}$$

$\check{R}$ will be explained later.

The nonlinear kinematic hardening rule proposed by Armstrong and Frederickson (1966) and refined by Marquis (1979; see also Benallna and Marquis, 1987) is $\dot{\hat{\boldsymbol{\alpha}}} = k_1 \dot{\boldsymbol{\varepsilon}}^p - k_2 \|\dot{\boldsymbol{\varepsilon}}^p\| \hat{\boldsymbol{\alpha}}$. Equation (103) is regarded as a modification obeying Ziegler's (1959) proposal for mathematical convenience which was described in Section 4; whereas, eqn (103) is integrated as

$$\frac{\hat{\alpha} - \hat{\alpha}_0}{\dfrac{k_1}{k_2} \mp \hat{\alpha}_0} = \pm \left[1 - \exp\left\{ \mp k_2 \sqrt{\tfrac{3}{2}} (\varepsilon^p - \varepsilon_0^p) \right\}\right] \tag{106}$$

for the initial condition $\hat{\alpha} = \hat{\alpha}_0 : \varepsilon^p = \varepsilon_0^p$ in uniaxial loading, where the upper and the lower cases stand for tension and compression, respectively. Here, it is worthwhile to notice that all the equations (99), (102) and (106) of the internal variables S, F and $\hat{\boldsymbol{\alpha}}$ reduce to the same exponential form of plastic strain in the uniaxial loading. Besides, eqn (99) and (106) describe concisely the gradual saturations of translations of S and $\hat{\boldsymbol{\alpha}}$ during a monotonic loading and the abrupt recovery of translations at the onset of stress reversal, which are typical properties of irreversible deformation.

The hardening of metals is interpreted to be caused by the accumulated plastic strain. For a certain accumulated plastic strain, the expansion of the yield surface, i.e. the so-called isotropic hardening during a cyclic loading is far more weakened than that during a monotonic loading and it saturates for a rather small accumulated plastic strain. In order to describe a cyclic loading behavior of metals, Chaboche *et al.* (1979) proposed the concept of (isotropically) nonhardening domain which means that the isotropic hardening does not occur when a plastic strain lies inside a certain surface in a plastic strain space. Let this surface be called a "hardening surface". It is similar to the conventional yield surface in a stress space, for which it is assumed that a plastic deformation occurs only when a stress lies on the surface. Based on this idea, they described the hardening surface by the equation

$$\sqrt{2/3}\,\|\breve{\boldsymbol{\varepsilon}}^p\| - K = 0 \tag{107}$$

where

$$\breve{\boldsymbol{\varepsilon}}^p \equiv \boldsymbol{\varepsilon}^p - \breve{\boldsymbol{\alpha}}. \tag{108}$$

Let the rate of translation and expansion of the hardening surface be given as

$$\dot{\breve{\boldsymbol{\alpha}}} \equiv (1-b)\breve{R}^{\zeta}\,\mathrm{tr}\,(\breve{\mathbf{n}}\dot{\boldsymbol{\varepsilon}}^p)\breve{\mathbf{n}} \tag{109}$$

$$\dot{K} \equiv \sqrt{2/3}\,b\breve{R}^{\zeta}\,\mathrm{tr}\,(\breve{\mathbf{n}}\dot{\boldsymbol{\varepsilon}}^p) \tag{110}$$

where b and ζ are material constants and

$$\breve{\mathbf{n}} \equiv \breve{\boldsymbol{\varepsilon}}^p/\|\breve{\boldsymbol{\varepsilon}}^p\| \tag{111}$$

$$\breve{R} \equiv \sqrt{2/3}\,\|\breve{\boldsymbol{\varepsilon}}^p\|/K. \tag{112}$$

Equations (109) and (110) satisfy eqn (107) when $\breve{R} = 1$. Equations (107)–(111) for $\breve{R}^{\zeta} = \Gamma$ were proposed by Chaboche *et al.* (1979) as $b = 1/2$ and modified by Ohno (1982) as $b \neq 1/2$, where Γ is defined as

$$\Gamma = 1 \quad \text{when}\ \sqrt{2/3}\,\|\breve{\boldsymbol{\varepsilon}}^p\| - K = 0 \quad \text{and} \quad \mathrm{tr}\,(\breve{\mathbf{n}}\dot{\boldsymbol{\varepsilon}}^p) > 0$$

$$\Gamma = 0 \quad \text{when}\ \sqrt{2/3}\,\|\breve{\boldsymbol{\varepsilon}}^p\| - K < 0 \quad \text{or} \quad \mathrm{tr}\,(\breve{\mathbf{n}}\dot{\boldsymbol{\varepsilon}}^p) \leqslant 0.$$

However, this hardening criterion falls within the framework of conventional plasticity, postulating that the interior of the hardening surface is a purely nonhardening domain. Thus, it would not be applicable to the cyclic loading behavior for plastic strain amplitudes which change within the hardening surface. Further, a discontinuous stress rate–strain rate relation is described with an abrupt occurrence of hardening when a plastic strain reaches this surface. In order to extend the concept of nonhardening domain to the unconventional plasticity framework, the new variable $\breve{R}$ is here incorporated into eqn (104) so that a hardening occurs depending on the value of $\breve{R}$ even when a plastic strain lies inside the hardening surface, and thus the continuous hardening rate is described always. Besides, the judgement whether the current plastic strain lies on the hardening surface or not is not required. By this extension or its refinement, a cyclic loading behavior of metals will be predicted properly for various stress or strain amplitudes.

$\dot{\mathbf{S}}$, $\dot{R}$ and $\dot{\boldsymbol{\varepsilon}}^p$ are given by eqns (48), (52) with eqn (58) and (61)–(63) themselves.

The subloading surface for the normal-yield surface (101) is expressed in the form of eqn (32) as

$$\sqrt{3/2}\,\|\tilde{\boldsymbol{\sigma}}' + R\hat{\mathbf{S}}'\| = RF \tag{113}$$

where

$$\tilde{\boldsymbol{\sigma}}' \equiv \tilde{\boldsymbol{\sigma}} - \tfrac{1}{3}\mathrm{tr}\,\tilde{\boldsymbol{\sigma}}\mathbf{I}, \quad \hat{\mathbf{S}}' \equiv \hat{\mathbf{S}} - \tfrac{1}{3}\mathrm{tr}\,\hat{\mathbf{S}}\mathbf{I}. \tag{114}$$

Solving eqn (113) for R, one obtains the analytical solution

$$R = \{J + \sqrt{(J^2 + Q\|\tilde{\boldsymbol{\sigma}}'\|^2)}\}/Q \tag{115}$$

where

$$J \equiv \mathrm{tr}\,(\tilde{\boldsymbol{\sigma}}'\hat{\mathbf{S}}'), \quad Q \equiv \tfrac{2}{3}F^2 - \|\hat{\mathbf{S}}'\|^2. \tag{116}$$

Equation (115) is the expression of R by the current stress $\boldsymbol{\sigma}$ and internal variables F, $\hat{\boldsymbol{\alpha}}$ and $\mathbf{S}$. Further, substituting eqn (115) into eqn (33), one has the expression for $\bar{\boldsymbol{\alpha}}$ by them.

Let an elastic constitutive equation be given by Hooke's law:

$$\dot{\boldsymbol{\varepsilon}}^e = \frac{1}{2G}\dot{\boldsymbol{\sigma}} - \left(\frac{1}{2G} - \frac{1}{E}\right)\mathrm{tr}\,\dot{\boldsymbol{\sigma}}\mathbf{I} \tag{117}$$

where E and G are Young's modulus and a shear modulus, respectively.

Figure 7 shows a relationship between $\sqrt{3}\tau$ and $\gamma^p/\sqrt{3}$ in a simple torsional cyclic loading measured by Tanaka *et al.* (1985), where τ and γ^p are the shear stress and twice the plastic shear strain, respectively (the equivalent stress $\sqrt{3/2}\,\|\boldsymbol{\sigma}'\|$ and the equivalent plastic strain rate $\sqrt{2/3}\,\|\dot{\boldsymbol{\varepsilon}}^p\|$ reduce to $\sqrt{3}|\tau|$ and $|\dot{\gamma}^p|/\sqrt{3}$ for a simple torsion and to $|\sigma|$ and $|\dot{\varepsilon}^p|$ for a uniaxial loading, respectively, where $\boldsymbol{\sigma}'$ and $\dot{\boldsymbol{\varepsilon}}^{p\prime}$ are a deviatoric stress and a deviatoric plastic strain rate). The thin-walled tubular specimen ($21^{+0.02}_{-0}$ mm diameter, $1^{+0.02}_{-0}$ mm thickness and 60 mm length) of type 316 stainless steel was subjected to cyclic loadings of plastic strain amplitudes of $\gamma^p/\sqrt{3} = \pm 0.1$, ± 0.2 and $\pm 0.4\%$. The cyclic loading in each stage—in each specified value of amplitude—was continued until the stabilized behavior was almost attained; the corresponding equivalent plastic strain $\int|\mathrm{d}\gamma^p|/\sqrt{3}$ was more than 30%.

The theoretical curves calculated by the present model are depicted in Fig. 8(a) where material constants and initial values are selected as follows:

$$h_1 = 1.5, \quad h_2 = 25, \quad k_1 = 15{,}000\ \mathrm{MPa}, \quad k_2 = 200$$
$$C = 700, \quad \chi = 0.9, \quad u_1 = 6{,}000, \quad m = 0.1$$
$$v = 5, \quad \zeta = 5, \quad b = 0.1$$
$$E = 199{,}000\ \mathrm{MPa}, \quad G = 77{,}000\ \mathrm{MPa}$$
$$F_0 = 250\ \mathrm{MPa}, \quad K_0 = 0, \quad \hat{\boldsymbol{\alpha}}_0 = \mathbf{S}_0 = \bar{\boldsymbol{\alpha}}_0 = \mathbf{0}$$

where K_0, $\hat{\boldsymbol{\alpha}}_0$, $\mathbf{S}_0$ and $\bar{\boldsymbol{\alpha}}_0$ are initial values of K, $\hat{\boldsymbol{\alpha}}$, $\mathbf{S}$ and $\bar{\boldsymbol{\alpha}}$, respectively. Variations of $\sqrt{3}\hat{\alpha}_\tau$, $\sqrt{3}(\hat{\alpha}_\tau + F\dot{\gamma}^p/|\dot{\gamma}^p|)$ (normal-yield state) and $\sqrt{3}S_\tau$ are depicted in Fig. 8(b), where $\hat{\alpha}_\tau$ and S_τ are shear components of $\hat{\boldsymbol{\alpha}}$ and $\mathbf{S}$. The smooth elastic–plastic transition, Masing effect and the saturation of hardening are shown in Fig. 8.

A good agreement between experiment and theory is observed in Figs 7 and 8(a).

8. DISCUSSIONS

The multi surface model proposed by Mróz (1966, 1967) and Iwan (1967) as an extension of the kinematic hardening model to the subyield state and the two surface model

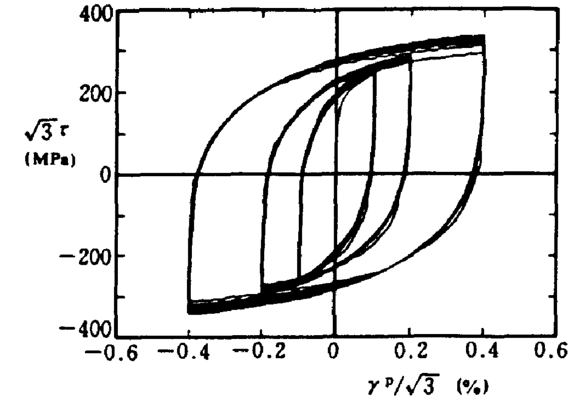

Fig. 7. Measured relationship of $\sqrt{3}\tau$ versus $\gamma^p/\sqrt{3}$ for simple torsional cyclic loadings of various plastic strain amplitudes (Tanaka *et al.*, 1985).

proposed by Dafalias and Popov (1975) and Krieg (1975) as a simplification of the multi surface model are well known and have been widely used as well as the initial subloading surface model or the bounding surface model with a radial mapping.

The unconventional models as the extention of the kinematic hardening model to the subyield state, i.e. the multi and the two surface models premise on the contact of loading surfaces, which is avoided exactly in the present models by the assumption [II']. It leads to the singularity of the field of hardening moduli in the contact point which is a similarity-center of the surfaces. Thus, a discontinuous stress rate–strain rate relation is described when a stress passes through the contact point after it left once from this point (in a

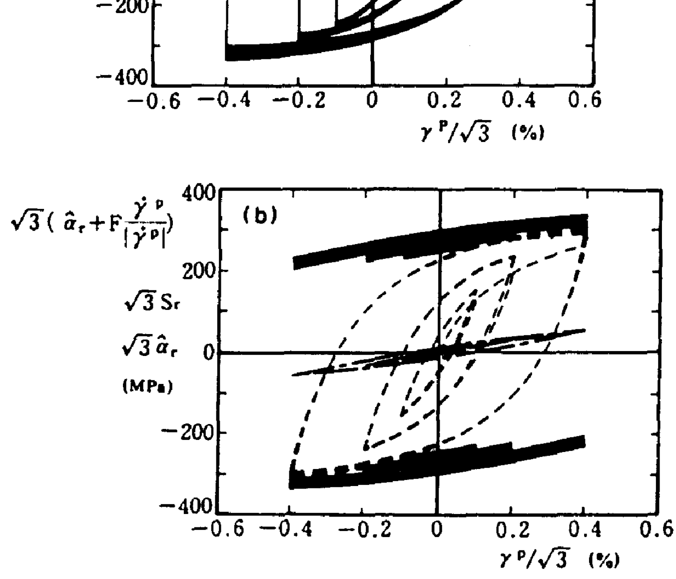

Fig. 8. Calculated relationship for simple torsional cyclic loadings of various plastic strain amplitudes: (a) $\sqrt{3}\tau$ versus $\gamma^p/\sqrt{3}$; (b) $\sqrt{3}\hat{\alpha}_\tau$ (— — —), $\sqrt{3}(\hat{\alpha}_\tau + F\dot{\gamma}^p/|\dot{\gamma}^p|)$ (——) and $\sqrt{3}S_\tau$ (- - - - -) versus $\gamma^p/\sqrt{3}$.

reloading after a partial unloading). Accordingly, these models are incapable of describing a smooth stress–strain curve in general. Further, the two surface model assumes the subyield surface enclosing an elastic domain. Then, even if there does not exist a contact of surfaces, a discontinuity of stress rate–strain rate relation occurs when a stress reaches the subyield surface.

Further, consider the uniaxial loading behavior of the material with the nonhardening von Mises normal-yield surface. The multi surface model predicts asymmetric unloading-reloading curves where the ending point of hysteresis loop coincides with its beginning point since the movements of nesting surfaces in the reloading state are just the inverse of those in the unloading state. Then, an unrealistically large hysteresis loop is predicted with the rounding coefficient $L = 2$ (Masing rule) and a mechanical ratchetting effect cannot be described tracing a fixed hysteresis loop cyclically. On the other hand, in the two surface model a plastic modulus is prescribed by the distance from a current stress point on the subyield surface to a conjugate stress on the normal-yield surface. Then, all the stress–strain curves are of same shape except for elastic parts. Accordingly, the Masing effect cannot be described by the two surface model, i.e. $L = 1$.

The multi surface model was mathematically refined to the infinite surface model by Mróz *et al.* (1981) assuming an infinite number of surfaces. It also involves, however, the above-mentioned shortcomings. Recently, in order to avoid them, Klisinski (1988) and Klisinski and Mróz (1988) proposed a method to remove a contact of the active loading and the stress reversal surfaces by combining a restrained translation of the former to the center of the latter and its expansion referring to the concept of the subloading surface model. However, this modification leads to a more peculiar problem that the two special lines are generated in the domain enclosed by the stress reversal surface, along which a stress rate induces a purely elastic deformation.

The two surface model was further simplified to the single surface model by Dafalias and Popov (1977) making the inner yield surface shrink to a point so that an elastic domain vanishes, and it was applied to metal by Dafalias (1977) and to concrete by Fardis *et al.* (1983). This model has the advantage to describe the mutual dependency between the directions of a stress rate and a plastic strain rate, while it is obviously incapable of describing the plastic deformation due to the stress change along the normal-yield surface. However, it is not applicable to the cyclic loading behavior predicting an excessive ratchetting with an open hysteresis loop as well as the initial subloading surface model. Also, it is incapable of describing a softening behavior at least by the existing formulation as was indicated by Hashiguchi (1985a, 1988). Furthermore, this model is accompanied with the mathematical inconvenience requiring analyses of nonlinear simultaneous equations in stress rate–strain rate analysis in general.

The subloading surface model does not assume a purely elastic domain. Also, when the normal-yield and the subloading surfaces come into contact, their sizes coincide with each other, i.e. $R = 1$, while the plastic modulus monotonically depends on NSR. Thus, a continuous stress rate–strain rate relation is described in a loading state, bringing about a smooth elastic–plastic transition. Further, the extended subloading surface model formulated in this paper is capable of describing fundamental plastic deformation behavior, i.e. an anisotropic hardening/softening and a hysteresis behavior including the Masing effect, a closed hysteresis loop and a mechanical ratchetting effect consistently by taking account of the movement of the similarity-center, i.e. the most elastic stress. Thus, it is to be applicable to the prediction of cyclic loading behavior in the subyield state, which has been required as the unconventional elastoplasticity.

The radial mapping model does not utilize the subloading surface and thus it uses the ratio of the magnitude of the current stress to that of the conjugate stress instead of NSR for the formulation of plastic strain rate equation by an interpolation method. These ratios are, however, mathematically the same when the subloading surface is similar to the normal-yield surface as is postulated in the initial or the present subloading surface model. Besides, the evolution equation of the projection-center is not formulated in a useful form and thus it is actually fixed. Dafalias (1986) and Klisinski (1988) assumed *a priori* the evolution equation of the similarity (projection or homology)-center which is the special form (isotropically and kinematically nonhardening normal-yield surface) of the equation proposed by the author (Hashiguchi, 1985b) and is inapplicable to deformation analysis of real materials with a hardening/softening. Thus, the concrete formulation of the radial mapping is regarded to lie in the stage of that of the initial subloading surface model. Besides, as was described in Section 4, the incorporation of the subloading surface is inevitable in order to formulate the extended consistency condition from which a plastic strain rate equation is derived reasonably, taking account of the physical requirement that NSR increases with a plastic deformation without the use of *ad hoc* method such as an interpolation. Further, the incorporation of the subloading surface would be unavoidable for further extension to a more generalized model which does not assume the similarity and to the tangential plasticity (Hashiguchi, 1989).

The subloading surface model as well as the other elastoplastic constitutive models without an adoption of such an *ad hoc* method as the intersection or the corner of plastic potential surfaces is incapable of describing the mutual dependency between the directions of a stress rate and a plastic strain rate and also the plastic deformation due to the stress change along the loading surface, however. Hereinafter, we should extend models so as to describe these behaviors also. Dafalias (1986) advocated the "hypoplasticity" referring to the hypoelasticity of Truesdell (1955). His definition of hypoplasticity or the naming of this term is not clear but he states that the hypoplastic constitutive equation includes a stress rate direction tensor, i.e. $\dot{\sigma}/\|\dot{\sigma}\|$ so that it becomes an incrementally nonlinear equation, while the hypoelasticity does not premise on the existence of a potential surface but excludes the nonlinearity. Thus, it requires the analysis of nonlinear simultaneous equations of stress rate strain rate. Obviously, it leads to a serious disadvantage to the analysis of boundary value problems. However, there does not exist the inevitable reason that one has to introduce the stress rate direction tensor directly in order to describe the mutual dependency between the directions of a stress rate and a plastic strain rate as is easily seen by the fact that even Hooke's law, the simplest linear equation, can describe a strain rate for any stress rate, and *vice versa*, their directions affecting each other. The further extension of the subloading surface model within the framework of the ordinary bilinear equation so as to describe the above-mentioned mutual dependency and the plastic deformation by the stress change along the loading surface will be exposed in a subsequent paper.

Acknowledgement—This paper was refined in the period that the author stayed in Institute of Fundamental Technological Research (IFTR), Warsaw, Poland. The author expresses his gratitude to Professor Z. Mróz in IFTR for his critical reading of this paper and valuable advice on clarity of exposition of the proposed model. Thanks are also due to Dr M. Ueno, Ryukyu University, Japan and Dr A. Jarzębowski in IFTR for their kind support on numerical calculations in this study.

REFERENCES

Aboim, C. A. and Wroth, C. P. (1982). Bounding-surface-plasticity theory applied to cyclic loading of sand. *Proc. 1st Int. Symp. Numer. Models Geomech.*, Zurich, p. 65.

Anandrajah, A. and Dafalias, Y. F. (1986). Bounding surface plasticity. III. Application to anisotropic cohesive soils. *J. Engng Mech. ASCE* **112**, 1292.

Armstrong, P. J. and Frederic, C. O. (1966). A mathematical representation of the multiaxial Baushinger effect. G.E.G.B. Report RD/B/N, p. 731.

Banerjee, S. and Pan, Y.-W. (1986). Transient yielding model for clays. *J. Engng Mech. ASCE* **112**, 170.

Bardet, J. P. (1986). A bounding surface plasticity model for sands, *J. Engng Mech. ASCE* **112**, 1198.

Benalla, A. and Marquis, D. (1987). Constitutive equations for nonproportional cyclic elastoplastic viscoplasticity. *J. Engng Technol. ASME* **109**, 326.

Chaboche, J. L., Dang-Van, K. and Cordier, G. (1979). Modelization of the strain memory effect on the cyclic hardening of 316 stainless steel. *Trans. 5th Int. Conf. SMiRT*, Berlin, Division L.

Chen, E.-S. and Buyukozturk, O. (1985). Constitutive model for concrete in cyclic compression. *J. Engng. Mech. ASCE* **111**, 797.

Dafalias, F. Y. (1981). A novel bounding surface constitutive law for the monotonic and cyclic hardening response of metals. *Trans. 6th Int. Conf. SMiRT*, Paris, L3/4.

Dafalias, Y. F. (1984). Modelling cyclic plasticity: simplicity versus sophistication. *Mechanical Engineering Materials*, p. 153. John Wiley.

Dafalias, Y. F. (1986). Bounding surface plasticity. I. Mathematical foundation and hypoplasticity. *J. Engng Mech. ASCE* **112**, 966.

Dafalias, Y. F. and Herrmann, L. R. (1980). A bounding surface soil plasticity. *Proc. Int. Symp. Soils Cyclic Trans. Load.*, Swansea, p. 335.

Dafalias, Y. F. and Herrmann, L. R. (1982). Bounding surface formulation of soil plasticity. *Soil Mech.—Trans. Cyclic Loads*, p. 253. John Wiley, New York.
Dafalias, Y. F. and Herrmann, L. R. (1986). Bounding surface plasticity. II: Application to isotropic cohesive soils. *J. Engng Mech. ASCE* **112**, 1263.
Dafalias, Y. F. and Popov, E. P. (1975). A model of nonlinearly hardening materials for complex loading. *Acta Mech.* **21**, 173.
Dafalias, Y. F. and Popov, E. P. (1976). Plastic internal variables formalism of cyclic plasticity. *J. Appl. Mech. ASME* **43**, 645.
Dafalias, Y. F. and Popov, E. P. (1977). Cyclic loading for materials with a vanishing elastic domain. *Nuclear Engng Design* **41**, 293.
Drucker, D. C. (1951). A more fundamental approach to plastic stress–strain relations. *Proc. 1st U.S. Natl. Congr. Appl. Mech. ASME* **1**, 487.
Drucker, D. C. (1988). Conventional and unconventional plastic response and representation. *Appl. Mech. Rev. ASME* **41**, 151.
Dungar, R. (1987). A viscoplastic bounding surface model for concrete under compressive and tensile conditions and its application to arch dam analysis. *Proc. 2nd Int. Conf. on Constitutive Laws for Engineering Mater.: Theory and Application*, Tucson, p. 849.
Edelman, F. and Drucker, D. C. (1951). Some extensions of elementally plasticity theory. *J. Franklin Inst.* **251**, 581.
Fardis, M. N., Alibe, B. and Tassoulas, L. (1983). Monotonic and cyclic constitutive laws for concrete. *J. Engng Mech. ASCE* **109**, 516.
Hashiguchi, K. and Ueno, M. (1977). Elastoplastic constitutive laws of soils. Constitutive Equations of Soils: *Proc. 9th ICSMFE*, Special Session 9, Tokyo, p. 73.
Hashiguchi, K. (1978). Plastic constitutive equations of granular materials. *Proc. US–Japan Seminar Continuum Mech. Statistical Approaches in Mech. Gran. Materials.*, Sendai, p. 321.
Hashiguchi, K. (1979). Constitutive equations of granular media with an anisotropic hardening. *Proc. 3rd Int. Conf. Numer. Meth. Geomech.*, Aachen, p. 435.
Hashiguchi, K. (1980a). Constitutive equations of elastoplastic materials with elastic–plastic transition. *J. Appl. Mech. ASME* **47**, 266.
Hashiguchi, K. (1980b). Anisotropic hardening model for granular media. *Proc. Int. Symp. Soils Cyclic Trans. Load.*, Swansea, p. 469.
Hashiguchi, K. (1981). Constitutive equations of elastoplastic materials with anisotropic hardening and elastic–plastic transition. *J. Appl. Mech. ASME* **48**, 297.
Hashiguchi, K. (1985a). Macrometric approaches—static intrinsically time-independent. Constitutive Laws of Soils: *Proc. 11th ICSMFE*, Discussion Session 1A, San Francisco, p. 25.
Hashiguchi, K. (1985b). Subloading surface model of plasticity. *Ibid.*, p. 127.
Hashiguchi, K. (1988). A mathematical modification of two surface model formulation in plasticity. *Int. J. Solids Structures* **24**, 987.
Hashiguchi, K. (1989). Subloading surface model with tangential plasticity. *Proc. 2nd Symp. Plasticity and its Applications*, Tsu, Japan (in print).
Herrmann, L. R., Kaliakin, V. A. M., Shen, C. K., Mish, K. D. and Zhu, Z.-Y. (1986). Numerical implementation of plasticity model for cohesive soils. *J. Engng Mech. ASCE* **113**, 500.
Hill, R. (1958). A general theory of uniqueness and stability in elastic-plastic solids. *J. Mech. Phys. Solids.* **6**, 236.
Hill, R. (1967). On the classical constitutive relations for elastic/plastic solids. *Recent Progress in Applied Mech.*, p. 241.
Ilyushin, A. A. (1961). O postulate plastichnosti (On the postulate of plasticity). *Prik. Math. Mek.* **25**, 503.
Ishlinski, I. U. (1954). Generalized theory of plasticity with linear strain hardening. *Ukr. Mat. Zh.* **6**, 314.
Iwan, W. D. (1967). On a class of models for the yielding behavior of continuous and composite systems. *J. Appl. Mech. ASME* **34**, 612.
Klisinski, M. (1988). Plasticity theory based on fuzzy set. *J. Engng Mech. ASCE* **114**, 563.
Klisinski, M. and Mróz, Z. (1988). Description of inelastic deformation and degradation of concrete. *Int. J. Solids Structures* **24**, 391.
Krieg, R. D. (1975). A practical two surface plasticity theory. *J. Appl. Mech. ASME* **42**, 641.
Liang, R. K. K., Sobhanie, M. and Timmermann, D. H. (1987). A bounding surface plasticity model for the stress–strain–time behavior of clays. *Proc. 2nd Int. Conf. Constitutive Laws Engng Mater.: Theory and Applications*, Tucson, p. 699.
Mair, G. and Hueckel, T. (1979). Nonassociated and coupled flow rules of elastoplasticity for rock-like materials. *Int. J. Rock Mech. Min. Sci. Geomech. Abstr.* **16**, 77.
Marquis, D. (1979). Sur un modèle de plasticité redant compte du comportement cyclique. *3ème Congr. Francais de Mecanique*, Nancy.
Masing, G. (1926). Eigenspannungen und Verfestigung beim Messing. *Proc. 2nd Int. Congr. Appl. Mech.*, Zurich, p. 332.
Mróz, Z. (1966). On forms of constitutive laws for elastic–plastic solids. *Archiwum Mechaniki Stosowanej* **18**, 3.
Mróz, Z. (1967). On the description of anisotropic workhardening. *J. Mech. Phys. Solids* **15**, 163.
Mróz, Z. and Norris, V. A. (1982). Elastoplastic and viscoplastic constitutive models for soils with anisotropic loading. *Soil Mech.—Trans. Cyclic Loads*, p. 305. John Wiley, New York.
Mróz, Z., Norris, V. A. and Zienkiewicz, O. C. (1981). An anisotropic, critical state model for soils subject to cyclic loading. *Geotechnique* **31**, 451.
Mróz, Z. and Zienkiewicz, O. C. (1984). Uniform formalization of constitutive equations for clays and sands. *Mechanical Engineering Materials*, p. 415. John Wiley, New York.
Naylor, D. J. (1985). A continuous plasticity versioin of the critical state model. *Int. J. Numer. Meth. Engng* **21**, 1187.
Ohno, N. (1982). A constitutive model of cyclic plasticity with a nonhardening strain region. *J. Appl. Mech. ASME* **49**, 721.

Pande, G. N. and Pietruszczak, St. (1982). Reflecting surface model for soils. *Proc. 1st Int. Symp. Numer. Models Geomech.*, Zurich, p. 50.

Pastor, M., Zienkiewicz, O. C. and Leung, K. H. (1985). Simple model for transient soil loading in earthquake analysis. II. Non-associated models for sands. *Int. J. Numer. Anal. Methods Geomech.* **9**, 477.

Pietruszczak, St. (1986). A flow theory of soil: Concept of multiple neutral loading surface. *Int. J. Comput. Geotech.* **2**, 185.

Prager, W. (1956). Recent development in the mathematical theory of plasticity. *J. Appl. Mech. ASME* **20**, 235.

Tanaka, E., Murakami, S. and Ooka, M. (1985). Effects of strain path shapes on nonproportional cyclic plasticity. *J. Mech. Phys. Solids* **33**, 559.

Truesdell, C. (1955). Hypo-elasticity. *J. Ratl. Mech. Anal.* **4**, 83.

Yang, B.-L., Dafalias, Y. F. and Herrmann, L. R. (1985). A bounding surface plasticity model for concrete. *J. Eng. Mech. ASCE* **111**, 359.

Ziegler, H. (1959). A modification of Prager's hardening rule. *Q. Appl. Phys.* **17**, 55.

Zienkiewicz, O. C., Leung, K. H. and Pastor, M. (1985). Simple model for transient soil loading in earthquake analysis. I. Basic model and its application. *Int. J. Numer. Anal. Methoids Geomech.* **9**, 453.

Zienkiewicz, O. C. and Mróz, Z. (1984). Generalized plasticity formulation and application to geomechanics. *Mech. Engineering Materials*, p. 655. John Wiley, New York.

Zienkiewicz, O. C. and Pastor, M. (1987). A general model for sand/soil behavior. *Proc. 2nd Int. Conf. Constitutive Laws Engng Mater.: Theory and Applications*, Tucson, p. 261.